

# INDISCRIMINATE COVERS OF INFINITE TRANSLATION SURFACES ARE INNOCENT, NOT DEVIOUS

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*The private lives of surfaces  
are innocent, not devious.  
- Kay Ryan*

**ABSTRACT.** We consider the interaction between passing to finite covers and ergodic properties of the straight-line flow on finite area translation surfaces with infinite topological type. Infinite type provides for a rich family of degree  $d$  covers for any integer  $d > 1$ . We give examples which demonstrate that passing to a finite cover can destroy ergodicity, but we also provide evidence that this phenomenon is rare. We define a natural notion of a random degree  $d$  cover and show that, in many cases, ergodicity and unique ergodicity are preserved under passing to random covers. This work provides a new context for exploring the relationship between recurrence of the Teichmüller flow and ergodic properties of the straight-line flow.

## 1. INTRODUCTION

A *translation surface* is a pair  $(S, \alpha)$ , where  $S$  is a Riemann surface and  $\alpha$  is a holomorphic 1-form on  $S$ . Let  $Z \subset S$  denote the zeros of  $\alpha$ . The 1-form  $\alpha$  endows  $S \setminus Z$  with local coordinates to the plane: for any  $p$  we have the locally defined *coordinate chart* to  $\mathbb{C}$  given by the local homeomorphism  $q \mapsto \int_p^q \alpha$ . These coordinate charts differ locally only by translation. A translation surface inherits a metric by pulling back the Euclidean metric on the plane along the coordinate charts. Points in  $S \setminus Z$  are locally isometric to the plane, while points in  $Z$  are cone singularities with cone angle  $2(k+1)\pi$  where  $k \geq 1$  is the degree of the zero of  $\alpha$ .

We'll say a translation surface is *classical* if  $S$  is a closed surface. Here, there is a well known interplay between two types of dynamical systems: (1) the dynamics of the *translation flow* on a translation surface  $(S, \alpha)$  given in local coordinates by

$$F^t : S \rightarrow S; \quad (x, y) \mapsto (x + t, y),$$

and (2) the Teichmüller deformation on the moduli space of translation surfaces, where  $g^t(S, \alpha)$  is obtained from  $(S, \alpha)$  by postcomposing the coordinate charts with the affine coordinate change  $g^t(x, y) = (e^{-t}x, e^ty)$ . Namely, the Teichmüller deformation renormalizes the translation flow. A famous consequence of this relationship is given by Masur's Criterion: if the forward orbit of the Teichmüller deformation,  $\{g^t(S, \alpha)\}_{t \geq 0}$ , has a convergent subsequence  $g^{t_n}(S, \alpha)$  with  $t_n \rightarrow \infty$  (i.e., the orbit is *non-divergent*), then the translation flow is uniquely ergodic [Mas92].

Recently, many results similar in spirit to Masur's criterion have been proven in special cases for translation surfaces of infinite genus. Due to the lack of a well defined moduli space of such surfaces, the standard approach is to define some topological space of surfaces where

$SL(2, \mathbb{R})$  acts and prove that non-divergence of an orbit  $t \mapsto g^t(S, \alpha)$  within this space entails consequences for the translation flow on  $(S, \alpha)$ . To date, the primary mechanism for building such a topological space of surfaces uses affine symmetries of the translation surface  $(S, \alpha)$ , as we will now explain.

Two translation surfaces  $(S, \alpha)$  and  $(S', \alpha')$  are *translation equivalent* if there is a homeomorphism  $h : S \rightarrow S'$  which is a translation in local coordinates. The group  $SL(2, \mathbb{R})$  acts on the collection of all translation surfaces by simultaneous postcomposition with all local charts; see §3. We define  $\mathcal{O}(S, \alpha)$  to be the  $SL(2, \mathbb{R})$  orbit:

$$\mathcal{O}(S, \alpha) = \{A(S, \alpha) : A \in SL(2, \mathbb{R})\} / \text{translation equivalence}.$$

The orbit  $\mathcal{O}(S, \alpha)$  is parameterized by a choice of  $A \in SL(2, \mathbb{R})$  and thus inherits a topology as a topological quotient space. This space can be described concretely as the quotient of  $SL(2, \mathbb{R})$  by a subgroup, namely the surface's *Veech group*,

$$V(S, \alpha) = \{A \in SL(2, \mathbb{R}) : A(S, \alpha) \text{ is translation equivalent to } (S, \alpha)\}.$$

Observe that two surfaces  $A_1(S, \alpha)$  and  $A_2(S, \alpha)$  with  $A_1, A_2 \in SL(2, \mathbb{R})$  are translation equivalent if and only if  $A_1$  and  $A_2$  determine the same left coset (or equivalently, if  $A_2^{-1}A_1 \in V(S, \alpha)$ ). Thus, there is a natural identification between  $\mathcal{O}(S, \alpha)$  and the left coset space  $SL(2, \mathbb{R})/V(S, \alpha)$ . In particular, this structure allows us to say that the Teichmüller trajectory  $g^t(S, \alpha)$  is *non-divergent in  $\mathcal{O}(S, \alpha)$*  if there is a sequence  $t_n \rightarrow +\infty$  so that  $g^{t_n}(S, \alpha)$  converges in  $\mathcal{O}(S, \alpha)$ .

Work of the second author [Tre14, Theorem 2] shows that if a finite area translation surface of possibly infinite genus  $(S, \alpha)$  has a non-divergent Teichmüller trajectory in  $\mathcal{O}(S, \alpha)$ , then the translation flow is ergodic on  $(S, \alpha)$ . (It seems reasonable to guess that in fact you get unique ergodicity in this case. A class of surfaces and directions for which this statement holds is provided in [Hoo10]. See Example F.2 and Appendix H of [Hoo10].)

Suppose  $(S, \alpha)$  is a finite area translation surface with infinite topological type and has ergodic translation flow. In this paper, we study the ergodicity of the translation flow on finite unbranched connected covers  $(\tilde{S}, \tilde{\alpha})$  of  $(S, \alpha)$ . In order to study this, we introduce some spaces of covers, and following the above paradigm study how the behavior of Teichmüller trajectories through this space influences the ergodicity of the translation flow.

In order to say something constructive, we pick some integer  $d \geq 2$ , and restrict attention to covers of degree  $d$ . Choose an arbitrary non-singular basepoint. Let  $(\tilde{S}, \tilde{\alpha})$  be a cover of  $(S, \alpha)$ , where the flat structure on  $(\tilde{S}, \tilde{\alpha})$  is lifted from the one on  $(S, \alpha)$ . The fiber of the basepoint can be identified in an arbitrary way with the set  $\{1, 2, \dots, d\}$ . The *monodromy action* is the natural right action of the fundamental group on fiber of the basepoint. Our identification of the fiber determines a *monodromy representation*  $\pi_1(S) \rightarrow \Pi_d$ , where  $\Pi_d$  is the permutation group of  $\{1, \dots, d\}$ . We note that a cover can be reconstructed from its monodromy representation, and two covers are isomorphic (in the sense of covering theory) if and only if these monodromy representations differ by conjugation by an element of  $\Pi_d$ . These ideas are reviewed in §4.1. Now fix a subgroup  $G \subset \Pi_d$ . We say a cover  $(\tilde{S}, \tilde{\alpha})$  has *monodromy in  $G$*  if it can be realized by a monodromy representation to  $G$ . From the above remarks, we note that the space of covers of  $(S, \alpha)$  with monodromy in  $G$  up to cover isomorphism is identified with

$$\Pi_d \backslash \text{Hom}(\pi_1(S), G).$$

We endow  $\text{Hom}(\pi_1(S), G)$  with the product topology by viewing it as a subset of  $G^{\pi_1(S)}$ , where the finite set  $G$  is given the discrete topology. In particular,  $\text{Hom}(\pi_1(S), G)$  is homeomorphic to a Cantor set, since  $\pi_1(S)$  is a free group with countably many generators.

We define  $\text{Cov}_G(S, \alpha)$  to be the collection of all covers of  $(S, \alpha)$  with monodromy in  $G$  up to translation equivalence. Translation equivalence is a coarser notion of equivalence than covering isomorphism, and we give  $\text{Cov}_G(S, \alpha)$  the quotient topology by viewing this space of covers as a topological quotient of  $\Pi_d \backslash \text{Hom}(\pi_1(S), G)$ . The choice of  $h \in \text{Hom}(\pi_1(S), G)$  determines a cover  $(\tilde{S}_h, \tilde{\alpha}_h)$  with  $h$  describing the monodromy of the cover. We note that the cover need not be connected. In fact  $(\tilde{S}_h, \tilde{\alpha}_h)$  is connected if and only if the image of the monodromy representation,  $h(\pi_1(S))$  acts transitively on  $\{1, \dots, d\}$ .

The main object of interest to this paper is the union of  $SL(2, \mathbb{R})$  orbits of covers of  $(S, \alpha)$  with monodromy in  $G$ :

$$(1) \quad \tilde{\mathcal{O}}_G(S, \alpha) = \{A(\tilde{S}, \tilde{\alpha}) : A \in SL(2, \mathbb{R}) \text{ and } (\tilde{S}, \tilde{\alpha}) \in \text{Cov}_G(S, \alpha)\} / \sim,$$

where  $\sim$  denotes translation equivalence. We call the diagonal action of  $g^t$  on  $\tilde{\mathcal{O}}_G(S, \alpha)$  the *cover cocycle*. Again,  $\tilde{\mathcal{O}}_G(S, \alpha)$  inherits a topology because it can be considered a topological quotient of  $SL(2, \mathbb{R}) \times \text{Cov}_G(S, \alpha)$ . This is a natural space in which to study Teichmüller deformations of covers of  $(S, \alpha)$ . Like  $\text{Cov}_G(S, \alpha)$ , this space contains both connected and disconnected surfaces provided  $G$  acts transitively on  $\{1, \dots, d\}$ . We prove the following:

**Theorem 1** (Connected accumulation point implies ergodicity). *Let  $d \geq 2$  be an integer and let  $G \subset \Pi_d$  be a subgroup which acts transitively on  $\{1, \dots, d\}$ . Let  $(S, \alpha)$  be a finite area translation surface with infinite topological type, and let  $(\tilde{S}, \tilde{\alpha})$  be a cover with monodromy in  $G$ . Then, the translation flow on  $(\tilde{S}, \tilde{\alpha})$  is ergodic if the Teichmüller trajectory  $g^t(\tilde{S}, \tilde{\alpha})$  has an  $\omega$ -limit point in  $\tilde{\mathcal{O}}_G(S, \alpha)$  representing a connected surface.*

Theorem 1 is proved in §5. We note that ergodicity of the translation flow on a finite cover has consequences for classifying the invariant measures.

**Proposition 2** (Ergodicity and unique lifts of measures). *Suppose that  $(\tilde{S}, \tilde{\alpha})$  is a finite cover of the finite area translation surface  $(S, \alpha)$  with infinite topological type. Also suppose that both surfaces have ergodic translation flow. Then, Lebesgue measure on  $(\tilde{S}, \tilde{\alpha})$  is the unique translation flow-invariant measure which projects to Lebesgue measure on  $(S, \alpha)$  under the covering map.*

See §6 for the brief proof.

**Corollary 3** (Lifting unique ergodicity). *Suppose the conditions of Proposition 2 are satisfied and additionally the translation flow on  $(S, \alpha)$  is uniquely ergodic. Then, the translation flow on the cover  $(\tilde{S}, \tilde{\alpha})$  is uniquely ergodic.*

*Proof.* Suppose  $\mu$  is a translation flow-invariant probability measure on  $(\tilde{S}, \tilde{\alpha})$ . Then its push forward under the covering map,  $p_*\mu$  is a translation-flow invariant measure on  $(S, \alpha)$ , and by uniqueness is necessarily Lebesgue measure. The proposition guarantees that  $\mu$  is also Lebesgue measure.  $\square$

The existence of a Veech group for the surface  $(S, \alpha)$  gives a mechanism for generating the type of accumulation we need for applying Theorem 1:

**Proposition 4.** *Suppose  $(S, \alpha)$  is a finite area translation surface with infinite topological type, and let  $(\tilde{S}, \tilde{\alpha})$  be a cover with monodromy in  $G$ . If  $(S, \alpha)$  has a non-divergent Teichmüller trajectory in  $\mathcal{O}(S, \alpha)$ , then the Teichmüller trajectory of the cover  $g^t(\tilde{S}, \tilde{\alpha})$  has an  $\omega$ -limit point in  $\tilde{\mathcal{O}}_G(S, \alpha)$ .*

Again, see §6 for the proof. In order to use the conclusions of Theorem 1, we would need to know that there is a connected  $\omega$ -limit point. We expect that it is difficult to determine precisely when this is true. However, we will introduce a notion under which most covers have connected accumulation points.

Let  $G \subset \Pi_d$  be as above, and let  $(S, \alpha)$  be a finite area translation surface with infinite topological type. We explain in §4.2 that there is a natural Borel probability measure  $m_G$  on the space  $\text{Cov}_G(S, \alpha)$  of covers with monodromy in  $G$  which is invariant under automorphisms of the base surface  $S$ . This gives us a notion of a “random cover.” Informally, the measure  $m_G$  corresponds to the notion of random cover obtained by flipping a fair coin to determine the images of the generators of the fundamental group under the monodromy representation.

**Theorem 5** (Random covers accumulate on connected covers). *Let  $(S, \alpha)$  be a finite area translation surface with infinite topological type, and suppose that it has a non-divergent Teichmüller trajectory in  $\mathcal{O}(S, \alpha)$ . Suppose  $G$  is a transitive subgroup of the permutation group  $\Pi_d$ . Then  $m_G$ -almost every cover  $(\tilde{S}, \tilde{\alpha}) \in \text{Cov}_G(S, \alpha)$  has a Teichmüller trajectory with a connected accumulation point.*

The proof lies in §6. As a consequence of Theorem 1, we see:

**Corollary 6** (Ergodicity of random covers). *With the hypotheses of Theorem 5,  $m_G$ -almost every cover of  $(S, \alpha)$  with monodromy in  $G$  has ergodic translation flow.*

Corollary 6 has applications to certain types of finite skew product extensions of  $n$ -adic odometers as follows. For any integer  $n \geq 2$ , let  $X_n = \{0, \dots, n-1\}^{\mathbb{N}}$  be given the product topology. The  $n$ -adic odometer is the map  $\Omega_n : X_n \rightarrow X_n$  defined as addition by 1 with infinite carry to the right:

$$\Omega_n : x = (x_1, x_2, \dots) \mapsto (0, 0, \dots, x_{\kappa(x)} + 1, x_{\kappa(x)+1}, \dots),$$

where  $\kappa(x)$  is the smallest index  $k$  so that  $x_k \neq n-1$ .

Denote by  $\Gamma$  the free group on countably infinite generators, which we will denote by  $\ell_i$  for  $i \in \mathbb{N}$ . Let  $n \geq 2$  be an integer and  $G$  a subgroup of  $\Pi_d$ . For any  $\psi \in \text{Hom}(\Gamma, G)$ , we define the  $\psi$ -skew product over the  $n$ -adic odometer  $E_\psi : X_n \times \{1, \dots, d\} \rightarrow X_n \times \{1, \dots, d\}$  as

$$(2) \quad E_\psi(x, m) = (\Omega_n(x), \psi(\ell_{k_x})(m)),$$

where  $k_x$  is the smallest index  $k$  so that  $x_k \neq 0$ .

For any  $n \geq 2$  and any subgroup  $G$  of  $\Pi_d$ , let  $\Lambda_{n,G}$  be the collection of all  $\psi$ -skew products over the  $n$ -adic odometer as above. This set has a natural product structure endowed with a product measure as follows. Since every element of  $\Lambda_{n,G}$  relies in a choice of an element in  $\text{Hom}(\Gamma, G)$ , the set  $\Lambda_{n,G}$  can be identified with the set  $\text{Hom}(\Gamma, G)$ , which comes with a natural product measure as in Definition 11 in §4.2. We will denote this measure by  $\mu_{n,G}$ .

**Theorem 7.** *Let  $n \geq 2$  be an integer and  $G$  be a transitive subgroup of  $\Pi_d$ . Then for  $\mu_{n,G}$ -almost every  $\psi \in \Lambda_{n,G}$ , the skew product  $E_\psi$  is uniquely ergodic.*

This theorem follows as a direct consequence of Corollaries 3 and 6, given the correct infinite translation surfaces. The surfaces used were first studied by Chamanara [Cha04]. We explain the proof at the beginning of §7.1.

The above approach relating accumulation of Teichmüller trajectories to connected covers to ergodic properties of the translation flow is just the beginning of the relationship. In §7, we work out some examples in more depth. One reason to do this is to produce examples where  $(S, \alpha)$  is a finite area translation surface with infinite topological type which is non-divergent in  $\mathcal{O}(S, \alpha)$  and hence has ergodic translation flow, but where  $(S, \alpha)$  has connected covers whose translation flow is not ergodic. We call such covers *devious*, because ergodicity is destroyed by passing to the cover. (This phenomenon does not occur for covers in the classical case, because we are using unbranched covers. If branching is allowed, ergodicity can be destroyed as first shown by Veech [Vee69].)

Devious covers must have interesting properties. Since  $(S, \alpha)$  is non-divergent in  $\mathcal{O}(S, \alpha)$ , and because the fibers of the projection  $p : \tilde{\mathcal{O}}_G(S, \alpha) \rightarrow \mathcal{O}(S, \alpha)$  are compact, there must be  $\omega$ -limit points of the trajectory  $g^t(\tilde{S}, \tilde{\alpha})$  in  $\tilde{\mathcal{O}}_G(S, \alpha)$ . But we can see from Theorem 1 that all such accumulation points must represent disconnected surfaces.

In §7, we produce some examples of devious covers. The examples of §7.1 concentrate on covers of the surface  $(S, \alpha)$  of Chamanara which is related to the 2-adic odometer. We show in Theorem 21 that for any non-trivial  $G$ , there is a dense set of covers with monodromy in  $G$  which are devious, i.e., translation flows on the dense set of covers are not ergodic. As noted above, such covers accumulate only on disconnected covers under Teichmüller flow, and it is natural to ask if there is a converse to this statement. In Theorem 22, we show that such a converse does not hold by producing covers of Chamanara's surface which only accumulate on disconnected covers under Teichmüller flow but nonetheless have ergodic translation flows.

The horizontal flow on Chamanara's surface is special for many reasons, including that the Teichmüller trajectory of this surface is periodic in  $\mathcal{O}(S, \alpha)$ , and so we consider a different surface in §7.2. We consider double covers of affine images of a translation surface we call the ladder surface,  $(S, \alpha)$ . The ladder surface has a non-elementary Veech group containing a parabolic and a reflection. Assuming that  $A(S, \alpha)$  recurs in  $\mathcal{O}(S, \alpha)$ , we see two possibly different behaviors for ergodic properties of covers. For some  $A \in SL(2, \mathbb{R})$ , we show every connected double cover has ergodic translation flow. For other  $A$ , we show that there are connected double covers with non-ergodic translation flow. Our covers can be thought to be determined by infinite strings of zeros and ones. When such non-ergodic covers exist, the choice of a non-ergodic cover has half the information content of a choice of a general connected cover. (See Theorem 39 and the comments below.) We associate a real number  $v$  to any geodesic  $g^t AV'$  through the quotient of  $SO(2) \backslash SL(2, \mathbb{R}) / V'$ , where  $V'$  is the known Veech group of  $(S, \alpha)$ . The multiplicative inverse  $1/v$  can be thought of as a combinatorial rate of escape of the geodesic lifted to a  $\mathbb{Z}$ -cover of the quotient  $SO(2) \backslash SL(2, \mathbb{R}) / V'$ . The real number  $v$  strongly distinguishes the two possible behaviors of double covers: If  $v > \varphi^2$ , where  $\varphi$  is the golden ratio, then every connected double cover of  $A(S, \alpha)$  has uniquely ergodic translation flow, but if  $v < \varphi^2$ , then we have non-ergodic connected double covers. This demonstrates the power of these methods to distinguish ergodic properties of the translation flow on finite covers from an understanding of Teichmüller trajectories in the space  $\tilde{\mathcal{O}}_G(S, \alpha)$ .

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DLGAPS 279893, and NSF Postdoctoral Fellowship DMS-1204008. The quote from on the first page is taken from *Surfaces* in [Rya97].

## 2. CONTEXT ON FINITE AREA FLAT SURFACES OF INFINITE TOPOLOGICAL TYPE

There has been an increased interest in the study of the dynamics and geometry of flat surfaces of infinite genus. Unlike compact flat surfaces of finite type, there is no natural space for parametrizing flat metrics for all surfaces of a given topological type. This gives the first obstacle to utilizing tools from the theory of compact surfaces. Different techniques have been developed to overcome this fundamental shortcoming, which prevents us from developing an theory to answer one of the most basic questions: whether the translation flow on a given flat surface of infinite type and finite area is ergodic or not.

Most studies concentrate on the case of the surface having finite or infinite area (a notably exception being [Hoo10], where the methods work for surfaces of finite and infinite area). Such choice has great implications to the tools used, the results obtained, and the method of construction used to produce examples or to define some “spaces of surfaces”. A common tool in both contexts is the use of Veech groups, which are a sort of symmetry groups of the surface. For compact flat surfaces, these are always discrete subgroups of  $SL(2, \mathbb{R})$ . Flat surfaces of finite type with non-trivial Veech groups are part of a very deep theory which has grown out of the foundational contributions of Thurston [Thu88] and Veech [Vee89], so the fact that they can be used in the infinite type setting is encouraging.

We will concentrate here on the development of the theory of flat surfaces of infinite type and finite area, though there is also a rapidly developing theory of flat surfaces of infinite type and infinite area (where often interest centers on infinite abelian covers of finite type flat surfaces).

To our knowledge, the first papers on dynamics on flat surfaces of infinite type are those which come out of the infinite step polygonal billiards introduced in [Tro99, DEDML98] through the unfolding procedure. All such surfaces considered were of finite area and came from “rational” polygons, i.e., the angles of the billiard from which the surfaces were constructed satisfied some rational conditions. Ergodic properties as well as topological results were obtained for a large class of these types of surfaces. The approach of these articles is very different than the approach here, because arguments do not make use of Veech groups.

The seminal paper of Chamanara [Cha04] introduced a 1-parameter family of flat surfaces of infinite type and of finite area with a non-trivial Veech group. The main results of that paper discussed the Veech groups that appear. Most importantly, even though the surfaces constructed possess many symmetries, the Veech group is never a lattice for any surface arising in this construction. We review Chamanara’s construction in section 7 and apply some of our results to spaces of covers thereof.

Another study of flat surfaces of infinite type and finite area has been the construction of Bowman which extends the Arnoux-Yoccoz family of flat surfaces to include a surface of infinite genus and finite area [Bow13]. This surface of infinite genus and finite area admits a pseudo-Anosov diffeomorphism. Moreover, it was found that the Veech group of this surface is isomorphic to  $\mathbb{Z} \times \mathbb{Z}_2$  and that the directions preserved by the pseudo-Anosov map correspond to uniquely ergodic translation flows.

In [Hoo10], a construction of Thurston is modified to produce infinite genus flat surfaces with non-elementary Veech groups. The construction sometimes produces finite area surfaces, and in this case the translation flow can be shown to be uniquely ergodic on various affine images of the surfaces.

The work [Tre14] addresses the question of ergodicity of translation flows for surfaces of finite area. The hypotheses of the main results are independent of topological type and therefore can be used to determine when the translation flow of a flat surface of infinite genus and finite area is ergodic or uniquely ergodic. We use this criterion in section 5 to study ergodic properties of translation flows on covers of infinite translation surfaces.

Finally, in [LT14] a way of constructing flat surfaces of finite area and infinite genus is developed through a connection to adic and cutting and stacking transformations, generalizing constructions of Bufetov in the classical case [Buf13]. Among other things, it is shown there that translation flows on surfaces of infinite genus and finite area can exhibit behavior which does not occur for translation flows on compact flat surfaces. In the present article, we also obtain results which are also not present for translation flows on compact surfaces, namely the existence of devious (unbranched) covers.

In general it is not well understood what Veech groups can arise for a flat surface of infinite type and finite area. In particular, it is unknown whether there exists a flat surface of infinite type and finite area whose Veech group is a lattice in  $SL(2, \mathbb{R})$ . (In contrast, it is known that for practically any subgroup  $G$  of  $SL(2, \mathbb{R})$ , there is a flat surface of infinite type and infinite area whose Veech group is  $G$  [PSV11].)

As it can be seen, Veech groups have played a significant role in the study of translation flows. The properties of translation flows so far obtained for surfaces of infinite genus and finite area are mostly similar to those of compact flat surfaces.

In this paper, we broaden the utilize structure provided by coverings to produce spaces of finite area flat surfaces of infinite type which are invariant under Teichmüller deformations, and provide sufficiently interesting Teichmüller dynamics so that varied phenomena appear for corresponding translation flows. This construction allows us to determine when the lift of a translation flow to a cover retains ergodic properties which the base surface possesses, such as unique ergodicity. This approach thus overcomes some of the problems created by not having a natural parameter space for infinite type flat surfaces.

### 3. BACKGROUND ON TRANSLATION SURFACES

Let  $(S, \alpha)$  be a translation surface. The *straight line flow* on  $(S, \alpha)$  in direction  $\theta$  is the flow  $F_\theta^t : S \rightarrow S$  defined for  $t \in \mathbb{R}$  given in local coordinates by  $F_\theta^t(x, y) = (x + t \cos \theta, y + t \sin \theta)$  away from the zeros of  $\alpha$ . We reserve the name *translation flow* for the straight line flow  $F_\theta^t$  where  $\theta = 0$ . This is the flow on  $S$  determined by the rightward unit vectorfield  $X$ . We will use  $Y$  denote the upward unit vectorfield.

A more global definition of the straight line flows can be done as follows. Since  $\alpha$  is holomorphic, the 1-forms  $\Re(\alpha)$  and  $\Im(\alpha)$  are harmonic, and thus closed. Therefore, the distributions  $\ker \Im(\alpha)$  and  $\ker \Re(\alpha)$  define a pair of foliations away from  $\Sigma$ , the *horizontal and vertical* foliations. The generators of the distributions in the unit tangent bundle of  $S - \Sigma$  are the vector fields  $X_\alpha$  and  $Y_\alpha$  which generate, respectively, the *horizontal and vertical* flows. The translation flow, as defined above, then corresponds to the horizontal flow. From this point of view, we will denote by  $\varphi_t^{X_\alpha}$  and  $\varphi_t^{Y_\alpha}$ , respectively, the horizontal and vertical flows.

There is a group of deformations of the flat metric on  $(S, \alpha)$  which is parametrized by the group  $SL(2, \mathbb{R})$ . This is the  $SL(2, \mathbb{R})$  action mentioned in the introduction. Fix a matrix  $A \in SL(2, \mathbb{R})$ . We get new (non-conformal) local coordinates to the plane by postcomposing the charts on  $(S, \alpha)$  to  $\mathbb{C}$  by the real-linear map

$$A : \mathbb{C} \rightarrow \mathbb{C}; \quad x + iy \mapsto a_{1,1}x + a_{1,2}y + i(a_{2,1}x + a_{2,2}y).$$

Then, we get a new Riemann surface structure  $S'$  on  $S$  by pulling back the complex structure using these deformed charts, and the charts determine a new holomorphic 1-form  $\alpha'$  on  $S'$ . We define  $A(S, \alpha) = (S', \alpha')$ . The action of the rotation subgroup  $O(2, \mathbb{R})$  parametrizes the directional flows on a given surface  $(S, \alpha)$ : for  $\theta \in O(2, \mathbb{R}) \simeq S^1$ , the horizontal flow on  $\theta(S, \alpha)$  corresponds to the straight line flow on  $(S, \alpha)$  in direction  $\theta$ .

Let  $(S, \alpha)$  and  $(S', \alpha')$  be translation surfaces. We say they are *translation equivalent* if there is a homeomorphism  $h : S \rightarrow S'$  which is a locally a translation in all local coordinate charts. The *Veech group* of  $(S, \alpha)$  is the subgroup  $V(S, \alpha) \subset SL(2, \mathbb{R})$  of elements  $A \in SL(2, \mathbb{R})$  so that  $A(S, \alpha)$  and  $(S, \alpha)$  are translation equivalent. An *affine homeomorphism* from a translation surface  $(S, \alpha)$  to another translation surface  $(S', \alpha')$  is a homeomorphism  $\phi : S \rightarrow S'$  so that there is a matrix  $A \in SL(2, \mathbb{R})$  so that in local coordinates

$$\phi(z_2) - \phi(z_1) = A(z_2 - z_1).$$

The matrix  $D(\phi) = A$  is called the *derivative* of the affine homeomorphism. Observe that the statement that  $A(S, \alpha)$  and  $(S', \alpha')$  are translation equivalent is equivalent to the statement that there is a affine homeomorphism  $\phi : (S, \alpha) \rightarrow (S', \alpha')$  with derivative  $A$ . The *affine automorphism group* of  $(S, \alpha)$ ,  $Aff(S, \alpha)$  is the group of all affine homeomorphisms from  $(S, \alpha)$  to itself. By the prior observation, we have  $D(Aff(S, \alpha)) = V(S, \alpha)$ .

#### 4. FINITE COVERS OF INFINITE SURFACES

In this section, we work out the theory of spaces of finite degree covers of an translation surface  $(S, \alpha)$  of infinite topological type. We describe the topology of the space of covers  $\text{Cov}_G(S, \alpha)$ , mentioned in the introduction, in subsection 4.1. In subsection 4.2, we place a natural Borel measure on this space. Subsection 4.3 discusses why disconnected covers should be considered rare.

**4.1. Spaces of covers.** Let  $S$  be a topological surface of infinite topological type. Choose a nonsingular basepoint  $s_0 \in S$ . The fundamental group  $\pi_1(S, s_0)$  is isomorphic to the free group with countably many generators.

A *covering* of  $(S, \alpha)$  is a pair  $(p, \tilde{S})$ , where  $\tilde{S}$  is a topological surface and  $p : \tilde{S} \rightarrow S$  is a topological covering. Two covers  $(p_1, \tilde{S}_1)$  and  $(p_2, \tilde{S}_2)$  are *isomorphic* if there is a homeomorphism  $h : \tilde{S}_1 \rightarrow \tilde{S}_2$  so that  $p_1 = p_2 \circ h$ .

We recall idea of the monodromy action from covering space theory. Let  $p : \tilde{S} \rightarrow S$  be a covering map. The *monodromy action* is the right action on the fiber over the basepoint,  $p^{-1}(s_0)$ , defined by

$$p^{-1}(s_0) \times \pi_1(S, s_0) \rightarrow p^{-1}(s_0); \quad \tilde{s} \cdot \gamma \mapsto \tilde{\beta}(1),$$

where  $\beta : [0, 1] \rightarrow S$  is a loop in the class of  $\gamma \in \pi_1(S, s_0)$ , and  $\tilde{\beta} : [0, 1] \rightarrow \tilde{S}$  is a lift (i.e.,  $p \circ \tilde{\beta} = \beta$ ) so that  $\tilde{\beta}(0) = \tilde{s}$ . It should be noted that the definition of  $\tilde{s} \cdot \gamma$  is independent of the choice of  $\beta$ . (Once  $\beta$  is chosen, its lift  $\tilde{\beta}$  is determined based on the condition that  $\tilde{\beta}(0) = \tilde{s}$ , and  $\tilde{\beta}(1)$  depends only on  $\gamma$ .)



It is a basic observation from covering space theory that the monodromy action determines the cover up to isomorphism. This includes disconnected covers of the connected surface  $S$ . We will briefly how to build a cover from an action. Concretely, given any right action of  $\pi_1(S, s_0)$  on a discrete set  $J$ , we can build a cover of  $S$  with this action as the monodromy action. To see this, fix such an action. For each  $j \in J$ , let  $\text{Stab}(j) \subset \pi_1(S, s_0)$  be the stabilizer of  $j$ . We can then build a cover  $\tilde{S}_j$  as the quotient of the universal cover of  $S$  by  $\text{Stab}(j)$ . It may be observed that by taking the disjoint union of such surfaces over all orbits  $[j]$  under the action, we obtain a cover,

$$(3) \quad \tilde{S} = \bigsqcup_{[j] \subset J} \tilde{S}_j$$

of  $S$  with the desired monodromy action. Furthermore, actions on two discrete sets  $J$  and  $J'$  determine isomorphic covers if and only if the actions are conjugate up to bijection, i.e., if there is a bijection  $f : J \rightarrow J'$  so that  $f(j \cdot \gamma) = f(j) \cdot \gamma$  for all  $\gamma \in \pi_1(S, s_0)$  and  $j \in J$ .

We will now specialize this discussion to finite covers of  $S$ . Suppose  $p : \tilde{S} \rightarrow S$  is a covering map of degree  $d$ . Make an arbitrary identification between the fiber  $p^{-1}(s_0)$  and the set  $\{1, \dots, d\}$ . Let  $\Pi_d$  be the symmetric group acting by permutations of  $\{1, \dots, d\}$ , and let

$$\ell : \{1, \dots, d\} \rightarrow p^{-1}(s_0)$$

be a labeling (a bijection) to the fiber. The associated *monodromy representation* (which depends on the labeling) is the group homomorphism  $M_\ell : \pi_1(S, s_0) \rightarrow \Pi_d$  defined so that

$$M_\ell(\gamma)(i) = \ell^{-1}(\ell(i) \cdot \gamma^{-1}) \quad \text{for all } \gamma \in \pi_1(S, s_0) \text{ and all } i \in \{1, \dots, d\}.$$

Conversely, such a representation determines an action on  $\{1, \dots, d\}$  and so, from the above discussion, a choice of a monodromy representation  $\pi_1(S, s_0) \rightarrow \Pi_d$  determines a  $d$ -fold cover of  $S$ . Given two such representations, the covers are isomorphic if and only if they differ by conjugation by an element of  $\Pi_d$ , which has the effect of changing the labeling function. Thus, the space of  $d$ -fold covers of  $S$  up to isomorphism is canonically identified with

$$\Pi_d \backslash \text{Hom}(\pi_1(S, s_0), \Pi_d), \quad \text{where } \Pi_d \text{ is acting by conjugation.}$$

We endow  $\text{Hom}(\pi_1(S, s_0), \Pi_d)$  with the topology of pointwise convergence (or equivalently, the subspace topology coming from the inclusion of  $\text{Hom}(\pi_1(S, s_0), \Pi_d)$  into the product space  $\Pi_d^{\pi_1(S, s_0)}$ ), and this space of covers gets the quotient topology.

As in the introduction if  $G$  is a subgroup of  $\Pi_d$  for an integer  $d \geq 2$ , we say that a cover  $\tilde{S}$  has *monodromy in  $G$*  if there is a representation  $\pi_1(S, s_0) \rightarrow G$  which determines a cover isomorphic to  $\tilde{S}$ . Note that this concept is independent of the basepoint. We define  $\text{TopCov}_G(S)$  to be the collection of all covers of  $(S, \alpha)$  with monodromy in  $G$  up to isomorphism of covers. Such covers are thus determined by elements of  $\text{Hom}(\pi_1(S, s_0), G) \subset \text{Hom}(\pi_1(S, s_0), \Pi_d)$ . So, the space  $\text{TopCov}_G(S)$  is in bijective correspondence with

$$(4) \quad \Pi_d \backslash \text{Hom}(\pi_1(S, s_0), G).$$

We use the identification with this quotient space to topologize  $\text{TopCov}_G(S, \alpha)$ .

**Proposition 8.** *The space  $\text{TopCov}_G(S, \alpha)$  is homeomorphic to a Cantor set.*

*Proof.* We use the characterization of Cantor sets as non-empty, perfect, compact, totally disconnected, and metrizable. We already know that  $\text{Hom}(\pi_1(S, s_0), G)$  is a Cantor set

and thus satisfies these properties. Because  $\text{TopCov}_G(S, \alpha)$  is a quotient of the Cantor set  $\text{Hom}(\pi_1(S, s_0), G)$ , we see it is non-empty and compact. Because the equivalence classes have finite size, we see that because  $\text{Hom}(\pi_1(S, s_0), G)$  is perfect, so is  $\text{TopCov}_G(S, \alpha)$ . Also because of this, we can use a metric on  $\text{Hom}(\pi_1(S, s_0), G)$  and use the Hausdorff metric restricted to equivalence classes to put a metric on  $\text{TopCov}_G(S, \alpha)$ . To see the space is totally disconnected, suppose  $h_1, h_2 \in \text{Hom}(\pi_1(S, s_0), G)$  have distinct images in  $\text{TopCov}_G(S, \alpha)$ . Then for any permutation  $p \in \Pi_d$ , there is an element  $\gamma_p \in \pi_1(S, s_0)$  so that  $h_1(\gamma_p) \neq p \cdot h_2(\gamma_p) \cdot p^{-1}$ . For each permutation  $q \in \Pi_d$ , consider the following sets  $U_q \subset \text{Hom}(\pi_1(S, s_0), G)$ :

$$U_q = \{h : h_1(\gamma_p) = q \cdot h(\gamma_p) \cdot q^{-1} \text{ for all } p \in \Pi_d\}.$$

The set  $U_q$  is clopen because it is a finite union of cylinder sets. (To see this, we write each  $\gamma_p$  as a product of generators. Then there are finitely many values  $h$  can take on the generators used so that each  $h(\gamma_p)$  equals  $q^{-1}h_1(\gamma_p)q$ .) Also observe  $h_1 \in U_e$ , where  $e \in \Pi_d$  is the identity element, and  $h_2 \notin U_q$  for any  $q$  from the remarks above. Thus the two complementary sets

$$U = \bigcup_{q \in \Pi_d} U_q \quad \text{and} \quad V = \bigcap_{q \in \Pi_d} \text{Hom}(\pi_1(S, s_0), G) \setminus U_q$$

are both clopen while  $h_1 \in U$  and  $h_2 \in V$ . Finally, we observe they are invariant under conjugation since

$$U = \bigcup_{q \in U_q} q^{-1}U_e q.$$

Thus  $U$  and  $V$  descend to clopen sets in  $\Pi_d \backslash \text{Hom}(\pi_1(S, s_0), G)$  which separate  $[h_1]$  and  $[h_2]$ , proving that this quotient is totally disconnected.  $\square$

Now suppose that we give the topological surface  $S$  a translation structure,  $(S, \alpha)$ . As in the introduction, we use  $\text{Cov}_G(S, \alpha)$  to denote the space of covers of  $(S, \alpha)$  with monodromy in  $G$  up to translation equivalence. Since isomorphic covers are translation equivalent,  $\text{Cov}_G(S, \alpha)$  is a topological quotient of  $\text{TopCov}_G(S, \alpha)$ .

**Remark 9.** *For many surfaces, isomorphism of covers is equivalent to translation equivalence. This happens whenever the quotient of the universal cover of  $(S, \alpha)$  by all its translation automorphisms is translation equivalent to  $(S, \alpha)$ . This is a form of geometric primitivity.*

Observe that a homeomorphism  $\phi : S \rightarrow S'$  between topological surfaces induces a homeomorphism between their spaces of covers with monodromy in  $G$  up to cover isomorphism. This is easily seen through the fundamental group. In order to consider their fundamental groups, we choose basepoints  $s_0$  and  $s'_0$ . To identify the fundamental groups, we make a choice of a curve  $\beta : [0, 1] \rightarrow S'$  so that  $\beta(0) = \phi(s_0)$  and  $\beta(1) = s'_0$ . The choice of  $\beta$  gives rise to a group homomorphism

$$(5) \quad \phi_\beta : \pi_1(S, s_0) \rightarrow \pi_1(S, s'_0); [\gamma] \mapsto [\beta^{-1} \bullet (\phi \circ \gamma) \bullet \beta],$$

where  $\bullet$  denotes path concatenation. This action allows  $\phi$  to act on the spaces of topological covers with monodromy in  $G$ . By identifying these spaces of covers as in equation 4, we see that this action is given by

$$(6) \quad \text{TopCov}_G(S) \rightarrow \text{TopCov}_G(S'); [h] \mapsto [h \circ \phi_\beta^{-1}],$$

where  $[h]$  is the  $\Pi_d$ -conjugacy class of a homomorphism  $h : \pi_1(S, s_0) \rightarrow G$ . Note that while the action of  $\phi$  on the fundamental group depended on the choice of the curve  $\beta$ , the action

on this space of covers does not, since the map  $h \mapsto h \circ \phi_\beta$  only changes by post-conjugation by a permutation when  $\beta$  is changed.

A homeomorphism  $(S, \alpha) \rightarrow (S', \alpha')$  between translation surfaces does not necessarily induce a homeomorphism from  $\text{Cov}_G(S, \alpha)$  to  $\text{Cov}_G(S', \alpha')$ . (This fails for instance, if  $(S, \alpha)$  admits translation automorphisms and  $(S', \alpha')$  does not.) However, an affine homeomorphism does induce such a homeomorphism. As above, this homeomorphism is induced by the map  $h \mapsto h \circ \phi_\beta^{-1}$ . We summarize this observation below.

**Proposition 10.** *Let  $\phi : (S, \alpha) \rightarrow (S', \alpha')$  be an affine homeomorphism with derivative  $A \in GL(2, \mathbb{R})$ , and let  $\phi_\beta : \pi_1(S, s_0) \rightarrow \pi_1(S, s'_0)$  be the group homomorphism as defined above for some choice of curve  $\beta$ . If  $(\tilde{S}, \tilde{\alpha})$  is a cover of  $(S, \alpha)$  with monodromy homomorphism  $h : \pi_1(S, s_0) \rightarrow G$ , then  $A(\tilde{S}, \tilde{\alpha})$  is translation equivalent to the cover of  $(S', \alpha')$  with monodromy homomorphism  $h \circ \phi_\beta^{-1}$ . Thus, map  $h \mapsto h \circ \phi_\beta^{-1}$  induces a homeomorphism  $A_* : \text{Cov}_G(S, \alpha) \rightarrow \text{Cov}_G(S', \alpha')$ , which depends only on the derivative  $A$  of the affine homeomorphism  $\phi$ .*

As a consequence of this proposition, we observe that the Veech group of  $(S, \alpha)$  acts on  $\text{Cov}_G(S, \alpha)$ .

**4.2. Measures on spaces of covers.** In this subsection, we will construct some natural measures on our spaces of covers. We begin by describing an abstract construction. Later in the subsection, we will specialize the discussion to our setting of translation surfaces.

Let  $\Gamma$  be the non-abelian free group with a countable generating set  $\{\gamma_i : i \in \mathbb{N}\}$ , and let  $G \subset \Pi_d$  as above. We endow the space  $\text{Hom}(\Gamma, G)$  with its natural product topology, which makes the  $\text{Hom}(\Gamma, G)$  homeomorphic to a Cantor set. This is the coarsest topology so that for each  $\eta \in \Gamma$  and each  $\sigma \in G$ , the set of the form

$$\{h : \text{Hom}(\Gamma, G) : h(\eta) = \sigma\}.$$

is open. A pair of collections,  $e_1, \dots, e_k \in \mathbb{N}$  and  $\sigma_1, \dots, \sigma_k \in G$ , determine a *cylinder set* in  $\text{Hom}(\Gamma, G)$ ,

$$(7) \quad \mathcal{C}(e_1, \dots, e_k; \sigma_1, \dots, \sigma_k) = \{h : \text{Hom}(\Gamma, G) : h(\gamma_{e_i}) = \sigma_i \text{ for } i = 1, \dots, k\}.$$

Each cylinder set is both closed and open in the product topology, and the collection of cylinder sets generate the topology.

To characterize a Borel measure on  $\text{Hom}(\Gamma, G)$ , it suffices to describe the measures of the cylinder sets.

**Definition 11.** The *product measure*  $\mu$  on  $\text{Hom}(\Gamma, G)$  is defined so that for every cylinder set we have

$$\mu(\mathcal{C}(e_1, \dots, e_k; \sigma_1, \dots, \sigma_k)) = \frac{1}{|G|^k}.$$

This is the product measure induced on  $\text{Hom}(\Gamma, G)$  by the counting measure on  $G$ .

We remark that this measure  $\mu$  is interesting even in the case when  $\Gamma$  is a finitely generated free group, and related questions remain open [Pud13].

Automorphisms of  $\Gamma$  act on  $\text{Hom}(\Gamma, G)$ . Concretely, if  $\phi : \Gamma \rightarrow \Gamma$  is an automorphism, then we can define

$$(8) \quad \phi_* : \text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma, G); \quad h \mapsto h \circ \phi^{-1}.$$

**Lemma 12.** *The action of any automorphism of  $\Gamma$  preserves the product measure  $\mu$  on  $\text{Hom}(\Gamma, G)$ . In particular, the measure  $\mu$  is independent of our choice of generating set.*

**Remark 13** (Proof in abelian case). *In the case when  $G$  is abelian,  $\text{Hom}(\Gamma, G)$  can be identified with the topological group  $G^{\mathbb{N}}$ , and  $\mu$  is Haar measure. In this case, the proposition follows from the naturality of Haar measure.*

*Proof.* Let  $\phi : \Gamma \rightarrow \Gamma$  be an automorphism, and let  $\phi_*$  be its action on  $\text{Hom}(\Gamma, G)$ :

$$\phi_* : \text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma, G); \quad h \mapsto h \circ \phi^{-1}.$$

We will prove that  $\phi_*$  preserves the product measure  $\mu$  on  $\text{Hom}(\Gamma, G)$ . It suffices to prove that the measures of cylinder sets are preserved. Let  $\mathcal{C} = \mathcal{C}(e_1, \dots, e_k; \sigma_1, \dots, \sigma_k)$  be a cylinder set. We will prove that  $\mu \circ \phi_*(\mathcal{C}) = 1/|G|^k$ .

Let  $X = \langle \gamma_{e_1}, \dots, \gamma_{e_k} \rangle \subset \Gamma$ . Since  $\Gamma = \langle \gamma_e : e \in \mathbb{N} \rangle$ , there is a finite set  $\{e'_1, \dots, e'_m\} \subset \mathbb{N}$  so that

$$\phi^{-1}(X) \subset \langle \gamma_{e'_1}, \dots, \gamma_{e'_m} \rangle.$$

We'll call the subgroup on the right hand side of the equation  $Y$ . By viewing  $\mu$  as the product of counting measures, we see

$$(9) \quad \mu \circ \phi^*(\mathcal{C}) = \frac{\#\{h \in \text{Hom}(Y, G) : h \circ \phi^{-1}(\gamma_{e_i}) = \sigma_i \text{ for } 1 \leq i \leq k\}}{|G|^m}.$$

So it suffices to show that the number of homomorphisms in the numerator is  $|G|^{m-k}$ .

As above, we can find a finite set  $\{e''_1, \dots, e''_n\} \subset \mathbb{N}$  so that

$$\phi(Y) \subset \langle \gamma_{e''_1}, \dots, \gamma_{e''_n} \rangle.$$

Call the set on the right hand side  $Z$ . Note that  $X \subset Z$ .

We recall some basic definitions from the theory of free groups. A *basis* of a free group  $F$  is a set  $x_1, \dots, x_k$  so that  $F = \langle x_1 \rangle * \dots * \langle x_k \rangle$ . A subgroup  $H$  of a free group  $F$  is a *free factor* if every (equivalently, some) basis of  $H$  can be extended to a basis of  $F$ .

Consider  $X$ ,  $Y$ , and  $Z$  as above. Observe that  $X$  is a free factor in  $Z$ . So,  $X$  is a free factor in  $\phi(Y)$  [Pud13, Claim 2.5]. That is, we can extend  $\{\gamma_{e_1}, \dots, \gamma_{e_k}\}$  to a free generating set of  $\phi(Y)$ . Using  $\phi^{-1}$ , we can pull this back to a generating set of  $Y$ . So, we have  $k \leq n$ , and there is a free generating set of  $Y$  given by  $\beta_1, \dots, \beta_m$  so that

$$\beta_i = \phi^{-1}(\gamma_{e_i}) \quad \text{for } 1 \leq i \leq k.$$

Since this set generates  $Y$ , we see that  $\text{Hom}(Y, G)$  is in bijective correspondence with the possible images of  $\{\beta_1, \dots, \beta_m\}$ . The last  $m - k$  elements in this basis are irrelevant to the values of  $\phi^{-1}(\gamma_{e_i})$ , so we see that there are exactly  $|G|^{m-k}$  possible values which give homomorphisms in the numerator of equation 9.  $\square$

Observe that whenever  $S$  is a topological surface of infinite topological type, then its fundamental group  $\pi_1(S, s_0)$  is isomorphic to the countably generated free group  $\Gamma$ . It is a simple observation that conjugation by a permutation preserves the measure  $\mu_G$  constructed above. By identifying  $\text{Hom}(\Gamma, G)$  with  $\text{Hom}(\pi_1(S, s_0), G)$  and  $\text{TopCov}_G(S)$  with  $\Pi_d \backslash \text{Hom}(\pi_1(S, s_0), G)$ , we obtain a measure  $\nu_G$  on  $\text{TopCov}_G(S)$ , which we call the *product measure* on  $\text{TopCov}_G(S)$ .

**Corollary 14.** *Let  $\phi : S \rightarrow S'$  be a homeomorphism between two topological surfaces of infinite topological type. Let  $G \subset \Pi_d$  for  $d \geq 2$ , and let  $\nu_G$  and  $\nu'_G$  be the product measures on  $\text{TopCov}_G(S)$  and  $\text{TopCov}_G(S')$ , respectively. Then,  $\nu'_G$  is the pushforward of the measure  $\nu_G$  under the map  $\text{TopCov}_G(S) \rightarrow \text{TopCov}_G(S')$  induced by  $\phi$  as in equation 6.*

*Proof.* We can identify each space of covers with  $\text{Hom}(\Gamma, G)$  using isomorphisms to the fundamental groups. As noted in equation 6, the action of a homeomorphism on the space of covers is induced by a group isomorphism between the fundamental groups and via our identifications, an automorphism of  $\Gamma$ . Lemma 12 tells us that the measure  $\mu_G$  is invariant under such automorphisms. Also the automorphism commutes with the (partially defined)  $\Pi_d$ -action. Thus, our measures  $\nu_G$  and  $\nu'_G$  are the same in view of the identification of each space of covers with  $\Pi_d \backslash \text{Hom}(\Gamma, G)$ .  $\square$

Now let  $(S, \alpha)$  be a translation surface of infinite topological type. Since  $\text{Cov}_G(S, \alpha)$  is a quotient of  $\text{TopCov}_G(S)$ , we obtain an measure  $m_G$  on  $\text{Cov}_G(S, \alpha)$  as the pushforward of  $\nu_G$ . We call  $m_G$  the *product measure* on  $\text{Cov}_G(S, \alpha)$ .

Recall that Proposition 10 says that a translation equivalence when  $A \in SL(2, \mathbb{R})$  and  $A(S, \alpha) = (S', \alpha')$ , there is an induced homeomorphism  $A_*$  from  $\text{Cov}_G(S, \alpha)$  to  $\text{Cov}_G(S', \alpha')$ . This homeomorphism respects the product measures on these spaces:

**Corollary 15** (Affine naturality of measures). *Let  $(S, \alpha)$  be a translation surface of infinite topological type. Let  $A \in SL(2, \mathbb{R})$  and let  $(S', \alpha') = A(S, \alpha)$ . Then  $m'_G = m_G \circ A_*^{-1}$  where  $m_G$  and  $m'_G$  are the product measures on  $\text{Cov}_G(S, \alpha)$  and  $\text{Cov}_G(S', \alpha')$ .*

*Proof.* Because  $A(S, \alpha) = (S', \alpha')$ , there must be an affine homeomorphism  $\phi : (S, \alpha) \rightarrow (S', \alpha')$  with derivative  $A$ . The homeomorphism  $A_* : \text{Cov}_G(S, \alpha) \rightarrow \text{Cov}_G(S', \alpha')$  lifts to a homeomorphism  $\Phi : \text{TopCov}_G(S, \alpha) \rightarrow \text{TopCov}_G(S', \alpha')$  by Proposition 10. Since  $\Phi_*(\nu_G) = \nu'_G$  by Corollary 14 and  $m_G$  and  $m'_G$  are obtained as images of  $\nu_G$  and  $\nu'_G$ , we see  $m'_G = m_G \circ A_*^{-1}$ .  $\square$

**4.3. Disconnected covers.** Let  $G \subset \Pi_d$  and let  $h \in \text{Hom}(\pi_1(S, s_0), G)$ . By interpreting  $h$  as the monodromy action of the fundamental group of  $S$  on the fibers of the basepoint, we obtain a cover  $\tilde{S}$  of  $S$  as in §4.1. This cover is explicitly described by equation (3), and we can see the following:

**Proposition 16.** *The cover associated to  $h \in \text{Hom}(\pi_1(S, s_0), G)$  is connected if and only if the image  $h(\pi_1(S, s_0))$  acts transitively on  $\{1, 2, \dots, d\}$ .*

In particular, in order to have connected covers of  $(S, \alpha)$  with monodromy in  $G$ , the subgroup  $G \subset \Pi_d$  must act transitively on  $\{1, 2, \dots, d\}$ . The goal of this subsection is to formulate the following precise version of the statement that the collection of all disconnected covers is small.

**Proposition 17.** *Let  $G \subset \Pi_d$  be a subgroup which acts transitively on  $\{1, 2, \dots, d\}$ . Let  $S$  be a topological surface of infinite topological type. Then,  $\nu_G$ -almost every cover in  $\text{TopCov}_G(S, \alpha)$  is connected.*

*Proof.* Let  $\mathcal{H}$  denote the collection of all subgroups of  $H \subset \Pi_d$  so that  $H$  does not act transitively on  $\{1, \dots, d\}$ , but so that  $H$  is conjugate to a subgroup of  $G$ . Note that  $\mathcal{H}$  is a finite set. Consider the set  $\mathcal{D} \subset \text{Cov}_G(S, \alpha)$  of disconnected covers with monodromy in  $G$ . Recall  $\text{Cov}_d(S, \alpha)$ , is a quotient of  $\text{Hom}(\pi_1(S, s_0), \Pi_d)$ . Let  $\tilde{\mathcal{D}} \subset \text{Hom}(\pi_1(S, s_0), \Pi_d)$  be the lift of  $\mathcal{D}$ . By definition of  $\nu_G$ , we have  $\nu_G(\mathcal{D}) = \mu_G(\tilde{\mathcal{D}})$ . Proposition 16 tells us that

$$\tilde{\mathcal{D}} = \bigcup_{H \in \mathcal{H}} \text{Hom}(\pi_1(S, s_0), H).$$

Let  $\tilde{\mathcal{D}}_H = \text{Hom}(\pi_1(S, s_0), H)$ . By subadditivity of measures, it suffices to prove that  $\mu_G(\tilde{\mathcal{D}}_H) = 0$  for all  $H \in \mathcal{H}$ .

Fix  $H \in \mathcal{H}$ . Observe that  $H$  is a proper subgroup of  $G$ , since  $H$  does not act transitively while  $G$  does. Fix some  $\epsilon > 0$ . We will show that  $\mu_G(\tilde{\mathcal{D}}_H) < \epsilon$ . Since  $H$  is a proper subset of  $G$ , we can find a  $k$  so that  $(\frac{|H|}{|G|})^k < \epsilon$ . Let  $\{e_i : i \in \mathbb{N}\}$  be a basis for  $\pi_1(S, s_0)$ . Observe that  $\tilde{\mathcal{D}}_H$  is contained in the union of cylinder sets

$$\bigcup_{(h_1, \dots, h_k) \in H^k} \mathcal{C}(e_1, \dots, e_k; h_1, \dots, h_k),$$

where we are using notation from equation 7. Observe that by monotonicity and by definition of  $\mu_G$ , we have

$$\mu_G(\tilde{\mathcal{D}}_H) \leq \sum_{(h_1, \dots, h_k) \in H^k} \mu_G(\mathcal{C}(e_1, \dots, e_k; h_1, \dots, h_k)) = \frac{|H|^k}{|G|^k} < \epsilon.$$

This proves that  $\mu_G(\tilde{\mathcal{D}}_H) = 0$ , and thus  $\nu_G(\mathcal{D}) = 0$  by the remarks in the previous paragraph.  $\square$

## 5. ERGODICITY

Let  $(S, \alpha)$  be a flat surface and let  $G$  be a subgroup of the permutation group  $\Pi_d$  for some integer  $d \geq 2$ . The group  $SL(2, \mathbb{R})$  acts on the the space of affine deformations of covers with monodromy in  $G$ ,  $\tilde{\mathcal{O}}_G(S, \alpha)$ , and the action of the diagonal subgroup,  $g^t$ , is the cover cocycle. (See 1.)

In this section, we prove Theorem 1, which pertains to a connected cover  $(\tilde{S}, \tilde{\alpha}) \in \text{Cov}_G(S, \alpha)$ : If the Teichmüller trajectory  $g^t(\tilde{S}, \tilde{\alpha})$  has an accumulation point in the covers cocycle  $\tilde{\mathcal{O}}_G(S, \alpha)$  representing a connected surface, then the translation flow on  $(\tilde{S}, \tilde{\alpha})$  is ergodic.

We will see that Theorem 1 is a consequence of the following result, which gives a criterion for ergodicity in terms of the geometries realized under the Teichmüller deformation.

**Theorem 18** ([Tre14]). *Let  $(S, \alpha)$  be a flat surface of finite area. Suppose that for any  $\eta > 0$  there exist a function  $t \mapsto \varepsilon(t) > 0$ , a one-parameter family of subsets*

$$S_{\varepsilon(t), t} = \bigsqcup_{i=1}^{C_t} S_t^i$$

*of  $S$  made up of  $C_t < \infty$  path-connected components, each homeomorphic to a closed orientable surface with boundary, and functions  $t \mapsto \mathcal{D}_t^i > 0$ , for  $1 \leq i \leq C_t$ , such that for*

$$\Gamma_t^{i,j} = \{\text{paths connecting } \partial S_t^i \text{ to } \partial S_t^j\}$$

*and*

$$(10) \quad \delta_t = \min_{i \neq j} \sup_{\gamma \in \Gamma_t^{i,j}} \text{dist}_t(\gamma, \Sigma)$$

*the following hold:*

- (1)  $\text{Area}(S \setminus S_{\varepsilon(t), t}) < \eta$  for all  $t > 0$ ,
- (2)  $\text{dist}_t(\partial S_{\varepsilon(t), t}, \Sigma) > \varepsilon(t)$  for all  $t > 0$ ,

- (3) the diameter of each  $S_t^i$ , measured with respect to the flat metric on  $(S, \alpha_t)$ , is bounded by  $\mathcal{D}_t^i$  and

$$(11) \quad \int_0^\infty \left( \varepsilon(t)^{-2} \sum_{i=1}^{C_t} \mathcal{D}_t^i + \frac{C_t - 1}{\delta_t} \right)^{-2} dt = +\infty.$$

Moreover, suppose the set of points whose translation trajectories leave every compact subset of  $S$  has zero measure. Then the translation flow is ergodic.

The theorem above is a geometric criterion for ergodicity. The spirit of the theorem is that if, as one deforms a flat surface  $(S, \alpha)$  using the Teichmüller deformation  $g^t$ , the geometry of the surface does not deteriorate too quickly (as measured by the diameter of big components, among other things), the translation flow is ergodic.

**Remark 19.** Without loss of generality, we can assume that any surface to which we apply Theorem 18 has area 1. The point is that if we can control the deforming geometry on a subset with measure arbitrarily close to the total measure of the surface (which is finite) then we obtain ergodicity.

Let us review the strategy of the proof of Theorem 1. The Teichmüller orbit of  $(S, \alpha)$  in  $\mathcal{O}(S, \alpha)$  corresponds to the Teichmüller deformation of the surface  $g^t(S, \alpha)$ . By assumption, it has a converging subsequence, i.e., there is an  $a \in SL(2, \mathbb{R})$  and a sequence of times  $t_k \rightarrow \infty$  such that  $g^{t_k}(S, \alpha) \rightarrow a(S, \alpha)$  in the sense of conformal structures. This means that for any neighborhood of the identity  $U \subset SL(2, \mathbb{R})$ , there is an  $r \in V(S, \alpha)$  and a  $k$  so that  $g^{t_k} r^{-1} a^{-1} \in U$ . Informally, this says we do not need to deform much to move from  $a(S, \alpha)$  to  $g^{t_k} r^{-1}(S, \alpha)$ , the latter of which is translation equivalent to  $g^{t_k}(S, \alpha)$ .

Through any subsequence  $t_k \rightarrow \infty$  as above we can control the geometry of the surface  $g^t(S, \alpha)$  at times nearly  $t_k$ . We also gain some control at these times over covers  $(\tilde{S}, \tilde{\alpha})$  with monodromy in  $G$  through the use of renormalizing elements  $r_k$  of the Veech group  $V(S, \alpha)$ . The Veech group acts on the spaces of covers, and we can control the geometry of the deformed cover  $g^t(\tilde{S}, \tilde{\alpha})$  by considering its convergence along a subsequence to a connected cover  $a(\tilde{S}^*, \tilde{\alpha}^*)$  in the cover cocycle  $\tilde{\mathcal{O}}_G(S, \alpha)$ . The convergence of  $g^{t_k}(S, \alpha)$  to  $a(S, \alpha)$  controls the deforming geometry of  $g^t(S, \alpha)$ , and we also have subsequential convergence in the fiber of covers of  $a(S, \alpha)$  to the connected cover  $a(\tilde{S}^*, \tilde{\alpha}^*)$ . Together these ideas allow us to control the geometry of  $g^t(\tilde{S}, \tilde{\alpha})$  for times  $t$  near each  $t_k$ . This control is sufficient to verify the hypothesis of Theorem 18 and conclude that the horizontal flow on  $(\tilde{S}, \tilde{\alpha})$  is ergodic.

*Proof of Theorem 1.* Let  $(S, \alpha)$  be a unit area translation surface with infinite topological type, and let  $(\tilde{S}, \tilde{\alpha})$  be a cover with monodromy in  $G \subset \Pi_d$ . We assume that  $g^t(\tilde{S}, \tilde{\alpha})$  has an  $\omega$ -limit point  $(\tilde{L}, \tilde{\beta}) = a(\tilde{S}^*, \tilde{\alpha}^*)$  in the cover cocycle  $\tilde{\mathcal{O}}(S, \alpha)$ , where  $(\tilde{S}^*, \tilde{\alpha}^*)$  is another cover with monodromy in  $G$  and  $a \in SL(2, \mathbb{R})$ . The limit point  $(\tilde{L}, \tilde{\beta})$  is a cover of the surface  $(L, \beta) = a(S, \alpha)$ . Recall that the Veech group  $V(S, \alpha)$  acts on the space of covers with monodromy in  $G$ ,  $\text{Cov}_G(S, \alpha)$ . By definition of the topology of  $\tilde{\mathcal{O}}_G(S, \alpha)$ , the existence of this  $\omega$ -limit point means we can find a sequence of times  $t_k \rightarrow \infty$ , a sequence of Veech group elements  $r_k$ , and a sequence  $w_k \in SL(2, \mathbb{R})$  tending to the identity such that  $g^{t_k} r_k^{-1} = w_k a$  and  $r_k(\tilde{S}, \tilde{\alpha}) \rightarrow (\tilde{S}^*, \tilde{\alpha}^*)$  in  $\text{Cov}_G(S, \alpha)$ . Here,  $w_k$  deforms the limit  $(L, \beta)$  into the approximate  $g^{t_k} r_k^{-1}(S, \alpha)$ , which is translation equivalent to  $g^{t_k}(S, \alpha)$ .

Recall that  $\Sigma$  denotes the points added in the metric completion of a surface, and define

$$L_\varepsilon := \{z \in L : \text{dist}_\beta(z, \Sigma) \geq \varepsilon\},$$

where the distance function  $\text{dist}_\beta(\cdot, \cdot)$  is defined with respect to the flat metric on  $L$  given by  $\beta$ .

Fix  $\eta > 0$ . The sets  $L_\varepsilon$  form an exhaustion of  $L$ , i.e., the sets are increasing as  $\varepsilon$  decreases and their union is all of  $L$ . Since  $L$  is connected, it follows that there is an  $\varepsilon_\eta > 0$  so that  $L_\varepsilon$  has a connected component of area at least  $1 - \eta$  when  $\varepsilon < \varepsilon_\eta$ . For  $\varepsilon < \varepsilon_\eta$ , let  $\epsilon \mapsto C_\epsilon$  be a nested family of choices of such a component. Then we can choose  $\varepsilon < \varepsilon' < \varepsilon_\eta$ , and find a connected compact subsurface  $L^\circ \subset L$  satisfying  $C_\varepsilon \supset L^\circ \supset C_{\varepsilon'}$ . By possibly removing some bits of this subsurface, we can arrange that  $L^\circ$  is a topological disk, contained in  $C_\varepsilon$  (but not necessarily containing  $C_{\varepsilon'}$ ), and that  $\text{Area } L^\circ > 1 - \eta$  is still satisfied. Let  $\mathcal{D}$  denote the diameter of the topological disk  $L^\circ$ .

Consider the cover  $(\tilde{L}, \tilde{\beta})$  of  $(L, \beta)$ , and let  $p_L : \tilde{L} \rightarrow L$  denote the covering map. Let  $\tilde{L}^\circ = p_L^{-1}(L^\circ)$ . Since  $L^\circ$  is a topological disk, the preimage  $\tilde{L}^\circ$  has  $d$  connected components, each of which is a copy of  $L^\circ$ . Select a basepoint  $\ell \in \tilde{L}^\circ$ , and choose a labeling of the lifts of the basepoints  $\{\ell_1, \dots, \ell_d\} = p_L^{-1}(\ell)$ . This allows us to describe our cover  $\tilde{L}$  as the cover determined by monodromy representation  $h^* : \pi_1(L, \ell) \rightarrow G$ . For each pair of distinct  $i, j \in \{1, \dots, d\}$ , choose a curve  $\gamma_{i,j} \subset \tilde{L}$  joining  $\ell_i$  to  $\ell_j$  which does not intersect  $\Sigma$ . Let  $\Gamma = \bigcup_{i,j} p_L(\gamma_{i,j}) \subset L$  and let

$$\delta = \min_{i,j} \text{dist}_{\tilde{\beta}}(\gamma_{i,j}, \Sigma) = \text{dist}_\beta(\Gamma, \Sigma).$$

Observe that  $\delta > 0$  since  $\Gamma$  is compact.

Observe that  $(L, \beta)$  is translation equivalent to  $w_k^{-1} g^{t_k}(S, \alpha)$  since we chose these group elements so that  $w_k^{-1} g^{t_k} r_k^{-1} = a$  and  $r_k \in V(S, \alpha)$ . We define

$$(\tilde{L}_k, \tilde{\beta}_k) = w_k^{-1} g^{t_k}(\tilde{S}, \tilde{\alpha}),$$

which is also a cover of  $(L, \beta)$ . Let  $p_k : \tilde{L}_k \rightarrow L$  be the covering maps, and define  $\tilde{L}_k^\circ = p_k^{-1}(L^\circ)$ . Again these surfaces have  $d$  components, and the normalized area satisfies

$$(12) \quad \text{Area}(\tilde{L}_k^\circ) \geq 1 - \eta.$$

Now recall that in  $\text{Cov}_G(S, \alpha)$ , we have  $r_k(\tilde{S}, \tilde{\alpha}) \rightarrow (\tilde{S}^*, \tilde{\alpha}^*)$ . It follows that

$$(13) \quad (\tilde{L}_k, \tilde{\beta}_k) = w_k^{-1} g^{t_k}(\tilde{S}, \tilde{\alpha}) \rightarrow (\tilde{L}, \tilde{\beta})$$

within  $\text{Cov}_G(L, \beta)$ . Since  $\text{Cov}_G(L, \beta)$  is a subspace of a topological quotient of the space of monodromy representations,  $\text{Hom}(\pi_1(L, \ell), \Pi_d)$ , we know that we can choose labels for the fibers for the basepoint  $p_k^{-1}(\ell) \subset \tilde{L}_k$  so that the induced representations to a conjugate of  $G$ ,  $h_k : \pi_1(L, \ell) \rightarrow \Pi_d$  converge in the product topology to  $h^*$ . This means that there is a  $K$  so that for  $k > K$ , we have

$$(14) \quad h_k(\gamma_{i,j}) = h^*(\gamma_{i,j}) \quad \text{for all distinct } i, j \in \{1, \dots, d\}.$$

We can label the  $d$  components of  $\tilde{L}_k^\circ$  according to the label of the basepoint each subsurface contains. We observe that (14) tells us that for each  $i \neq j$ , there is a lift of  $p_L(\gamma_{i,j})$  under the projection  $p_k : \tilde{L}_k \rightarrow L$  which joins the point in  $p_k^{-1}(\ell)$  labeled  $i$  to the the point labeled  $j$ . Taken together, these lifts give a collection of curves joining all pairs of components of



$\tilde{L}^\circ$ . We let  $\Gamma_k \subset \tilde{L}_k$  denote the union of lifted curves. These curves in  $\tilde{L}_k$  also get no closer than  $\delta$  to the completion locus  $\Sigma$ .

Now choose a flow box  $B$  about the identity for the flow  $a \mapsto g^t a$  in  $SL(2, \mathbb{R})$ . That is, we choose  $B \subset SL(2, \mathbb{R})$  to be a compact neighborhood of the origin so that whenever  $a \in B$ , the maximal interval,  $I(a)$ , containing zero and contained in the set

$$\{t \in \mathbb{R} : g^t a \in B\}$$

has length  $|I(a)| = \tau > 0$ . Recall that  $w_k = g^{t_k} r_k^{-1} a^{-1}$  converges to the identity, and so by possibly dropping the first few terms in our sequences, we can assume that each  $w_k$  lies in our flow box  $B$ . For each  $k$ , define

$$I_k = I(w_k) + t_k.$$

Then  $I_k$  is the maximal interval of times  $t$  contiguous with  $t_k$  so that  $g^t r_k^{-1} a^{-1} \in B$ . Since  $t_k \rightarrow \infty$  and the length of these intervals is always  $\tau$ , by passing to a subsequence, we can assume that the intervals  $I_k$  are all disjoint.

Our flow box  $B$  is compact, so we can find constants  $0 < m < 1 < M$  so that for any non-zero vector  $\mathbf{v} \in \mathbb{R}^2$ ,

$$(15) \quad 0 < m|\mathbf{v}| < |b(\mathbf{v})| < M|\mathbf{v}| \quad \text{for all } b \in B.$$

These constants control how metrics deform when move within the flow box.

We have quite explicit control of the geometry of the covers  $(\tilde{L}_k, \tilde{\beta}_k)$ . We have chosen a subsurface  $\tilde{L}_k^\circ$ , which is a union of  $d$  topological disks joined by curves which lie in the set  $\Gamma_k$ . The diameter of each of these components of  $\tilde{L}_k^\circ$  is always precisely  $\mathcal{D}$ , the distance from  $\tilde{L}_k^\circ$  to  $\Sigma$  is more than  $\epsilon$ , and the curves in  $\Gamma_k$  are always distance at least  $\delta$  from  $\Sigma$ .

Recalling the definition of  $(\tilde{L}_k, \tilde{\beta}_k)$  in (13), we see that  $g^{t_k}(\tilde{S}, \tilde{\alpha}) = w_k(\tilde{L}_k, \tilde{\beta}_k)$ . Then, for  $t \in I_k$ , we have  $g^{t-t_k} w_k \in B$  and

$$g^t(\tilde{S}, \tilde{\alpha}) = g^{t-t_k} w_k(\tilde{L}_k, \tilde{\beta}_k).$$

Since each  $g^{t-t_k} w_k \in B$ , using (15), we obtain control on the geometry of  $g^t(\tilde{S}, \tilde{\alpha})$ , through the subsurfaces  $\tilde{S}_t^\circ = g^{t-t_k} w_k(\tilde{L}_k^\circ)$  and curves in  $\Gamma_t = g^{t-t_k} w_k(\Gamma_k)$ . Namely, for  $t \in I_k$ ,

- The subsurface  $\tilde{S}_t^\circ$  has  $d$  components each with diameter no more than  $M\mathcal{D}$  measured in the metric of  $g^t(\tilde{S}, \tilde{\alpha})$ .
- The distance from  $\tilde{S}_t^\circ$  to  $\Sigma$  is more than  $\epsilon m$ .
- The minimal distance from the curves in  $\Gamma_t$  to  $\Sigma$  is more than  $\delta m$ .

Using this, we observe that the quantity being integrated in Theorem 18 is always at least as large as the constant

$$\left( (\epsilon m)^{-2} d(M\mathcal{D}) + \frac{d-1}{\delta m} \right)^{-2}.$$

Since this is a positive constant independent of  $k$ , and the total length of  $\bigcup_k I_k$  is infinite, we see that the integral in Theorem 18 is infinite. We conclude that the translation flow on  $(\tilde{S}, \tilde{\alpha})$  is ergodic with respect to Lebesgue measure on this cover.  $\square$

## 6. FURTHER PROOFS

In this section, we prove Proposition 2, Proposition 4 and Theorem 5 from the introduction.

*Proof of Proposition 2 (Ergodicity and unique lifts of measures).* Let  $(S, \alpha)$  be a translation surface, let  $(\tilde{S}, \tilde{\alpha})$  be a degree  $d$  cover, and let  $p : \tilde{S} \rightarrow S$  denote the covering map. We assume that the translation flow is ergodic on both of these surfaces. We will prove that Lebesgue measure is the unique translation flow invariant measure which projects to Lebesgue measure on  $(S, \alpha)$  under the covering map.

Fix a non-singular basepoint  $s \in S$ , and let  $h : \pi_1(S, s) \rightarrow \Pi_d$  be the monodromy representation. We will consider the regular (or normal) cover introduced by  $S$  associated to the subgroup  $\ker h \subset \pi_1(S, s)$ . Let  $(\hat{S}, \hat{\alpha})$  denote this cover. Because the subgroup  $\ker h$  is normal, there is a covering group action of  $\Delta = \pi_1(S, s)/\ker h$  on  $\hat{S}$ , and the quotient  $\hat{S}/\Delta$  is naturally identified with  $S$ . The covering  $\hat{S} \rightarrow S$  factors through  $\tilde{S}$ . That is, there is a subgroup  $\Gamma \subset \Delta$  so that the  $\tilde{S}$  is isomorphic as a cover to  $\hat{S}/\Gamma$ . We let  $\hat{p} : \hat{S} \rightarrow \tilde{S}$  denote the covering obtained by identifying  $\tilde{S}$  with  $\hat{S}/\Gamma$ .

Now suppose that  $\tilde{\mu}$  is a measure on  $\tilde{S}$  which is invariant for the translation flow, and satisfies  $p_*(\tilde{\mu}) = \lambda$ , where  $\lambda$  denotes Lebesgue measure on  $(S, \alpha)$ . Note that the measure  $\tilde{\mu}$  lifts to a unique measure  $\hat{\mu}$  on  $(\hat{S}, \hat{\alpha})$  so that  $\hat{p}_*(\hat{\mu}) = \tilde{\mu}$  and so that  $\gamma_*(\hat{\mu}) = \hat{\mu}$  for all  $\gamma \in \Gamma$ . Because  $\hat{\mu}$  projects through to Lebesgue measure on  $S$ , we know that if we average the push forwards of  $\hat{\mu}$  under the covering group  $\Delta$ , we get Lebesgue measure on  $(\hat{S}, \hat{\alpha})$ , which we denote by  $\hat{\lambda}$ . That is,

$$\frac{1}{|\Delta|} \sum_{\delta \in \Delta} \delta_*(\hat{\mu}) = \hat{\lambda}.$$

Now consider the push-forward of these measures under the covering map  $\hat{p}$ . Since  $p_*(\hat{\mu}) = \tilde{\mu}$ , we see that

$$\frac{1}{|\Delta|} \left( \tilde{\mu} + \sum_{\delta \in \Delta \setminus \{e\}} \hat{p}_* \circ \delta_*(\hat{\mu}) \right) = \tilde{\lambda},$$

where  $\tilde{\lambda}$  is the Lebesgue measure on  $(\tilde{S}, \tilde{\alpha})$ . Finally,  $\tilde{\lambda}$  is ergodic, so each of the probability measures in the above convex combination must equal  $\tilde{\lambda}$ . In particular,  $\tilde{\mu} = \tilde{\lambda}$ , which concludes the proof.  $\square$

*Proof of Proposition 4.* We assume that  $(S, \alpha)$  is a finite area translation surface with infinite topological type and that it has Teichmüller trajectory which is non-divergent in  $\mathcal{O}(S, \alpha)$ . By non-divergence, there is a subsequence of times  $t_n \rightarrow \infty$  so that  $g^{t_n}(S, \alpha)$  tends to some  $A_\infty(S, \alpha) \in \mathcal{O}(S, \alpha)$ , where  $A_\infty \in SL(2, \mathbb{R})$ . Because the topology on  $\mathcal{O}(S, \alpha)$  arises as a quotient of the topology on  $SL(2, \mathbb{R})$ , we see that there is a sequence  $A_n \in SL(2, \mathbb{R})$  tending to  $A_\infty$  so that  $g^{t_n}(S, \alpha)$  is translation equivalent to  $A_n(S, \alpha)$ . It then follows that there is a sequence of elements  $R_n$  of the Veech group of  $(S, \alpha)$  so that  $g^{t_n} = A_n R_n$ . Now consider a cover  $(\tilde{S}, \tilde{\alpha})$  with monodromy in  $G$ . Proposition 10 explains that the Veech group acts on the space of covers with monodromy in  $G$ . In particular, for each  $n$ , there is a cover  $(\tilde{S}_n, \tilde{\alpha}_n) \in \text{Cov}_G(S, \alpha)$  which is translation equivalent to  $R_n(\tilde{S}, \tilde{\alpha})$ . Then, in the space  $\tilde{\mathcal{O}}_G(S, \alpha)$ , we have that

$$g^{t_n}(\tilde{S}, \tilde{\alpha}) = A_n R_n(\tilde{S}, \tilde{\alpha}) = A_n(\tilde{S}_n, \tilde{\alpha}_n).$$

Now observe that  $\text{Cov}_G(S, \alpha)$  is a quotient of a Cantor set and thus sequentially compact, so there must be a limit point  $(\tilde{S}_\infty, \tilde{\alpha}_\infty)$  for the sequence  $(\tilde{S}_n, \tilde{\alpha}_n) \in \text{Cov}_G(S, \alpha)$ . Since  $A_n$  tends to  $A_\infty$  in  $SL(2, \mathbb{R})$ , we see that  $g^{t_n}(\tilde{S}, \tilde{\alpha})$  tends to  $A_\infty(\tilde{S}_\infty, \tilde{\alpha}_\infty)$ , which is our desired accumulation point.  $\square$

*Proof of Theorem 5 (Random covers accumulate on connected covers).* Let  $(S, \alpha)$  be a finite area translation surface with infinite topological type and a Teichmüller trajectory which is non-divergent in  $\mathcal{O}(S, \alpha)$ . As in the previous proof, this guarantees that there is a sequence  $t_n \rightarrow \infty$  so that  $g^{t_n} = A_n R_n$  where  $\{A_n \in SL(2, \mathbb{R})\}$  is a sequence tending to  $A_\infty \in SL(2, \mathbb{R})$ , and  $R_n \in V(S, \alpha)$ .

We will work with the space  $\text{TopCov}_G(S, \alpha)$  of topological covers of  $(S, \alpha)$  which comes equipped with a measure  $\nu_G$ . In order to do this, observe that for each Veech group element,  $R_n \in V(S, \alpha)$ , we can find an affine homeomorphism  $\phi_n : (S, \alpha) \rightarrow (S, \alpha)$  so that  $D(\phi_n) = R_n$ . With an additional choice of a curve for each  $n$ , we obtain an action

$$\Phi_n : \text{TopCov}_G(S) \rightarrow \text{TopCov}_G(S)$$

as in equation 6. When  $(\tilde{S}, \tilde{\alpha})$  is the translation surface obtained by lifting the structure from a topological covering  $(p, \tilde{S})$ , then  $R_n(\tilde{S}, \tilde{\alpha})$  is translation equivalent to the translation surface obtained by lifting the structure on  $\Phi_n(p, \tilde{S})$ . This follows from Proposition 10 and equation 6. In particular, with these hypotheses, we have that  $g^{t_n}(\tilde{S}, \tilde{\alpha})$  is translation equivalent to  $A_n \Phi_n(p, \tilde{S})$ , where  $\Phi_n(p, \tilde{S})$  is considered with the lifted translation structure. Since the sequence  $\{A_n\}$  converges in  $SL(2, \mathbb{R})$ , it suffices to find a connected accumulation point of the sequence  $\{\Phi_n(p, \tilde{S})\}$  for  $\nu_G$ -almost every  $(p, \tilde{S}) \in \text{TopCov}_G(S)$ . (This is because our measure  $m_G$  on the space of covers up to translation equivalence,  $\text{Cov}_G(S, \alpha)$ , is the push forward of  $\nu_G$  under the projection  $\text{TopCov}_G(S, \alpha) \rightarrow \text{Cov}_G(S, \alpha)$ .)

Let  $\mathcal{D} \subset \text{TopCov}_G(S)$  denote the collection of disconnected surfaces. Note that these surfaces have  $\nu_G$ -measure zero by Proposition 17. Let  $\mathcal{E} \subset \text{TopCov}_G(S)$  denote the collection of covers  $(p, \tilde{S}) \in \text{TopCov}_G(S)$  so that every accumulation point of  $\{\Phi_n(p, \tilde{S})\}$  is disconnected. Then if  $\mathcal{U} \subset \text{TopCov}_G(S)$  is open and contains  $\mathcal{D}$ , for every  $(p, \tilde{S}) \in \mathcal{E}$ , there must be an  $N$  so that  $\Phi_n(p, \tilde{S}) \in \mathcal{U}$  for all  $n > N$ . Indeed, if this were not true, then infinitely many  $\Phi_n(p, \tilde{S})$  lie in the complement of  $\mathcal{U}$ , which is sequentially compact since  $\text{Cov}_G(S, \alpha)$  is a Cantor set by Proposition 8. In other words, we have

$$\mathcal{E} \subset \bigcup_N \bigcap_{n > N} \Phi_n^{-1}(\mathcal{U}) = \liminf_{n \rightarrow \infty} \Phi_n^{-1}(\mathcal{U}).$$

Now fix some  $\epsilon > 0$ . We will show that  $\nu_G(\mathcal{E}) < \epsilon$ . Because  $\nu_G$  is a Borel probability measure on a Cantor set,  $\nu_G$  is regular. Thus because  $\nu_G(\mathcal{D}) = 0$ , we can find an open set  $\mathcal{U}$  containing  $\mathcal{D}$  so that  $\nu_G(\mathcal{U}) < \epsilon$ . Then from the above, we have

$$\nu_G(\mathcal{E}) \leq \nu_G \left( \liminf_{n \rightarrow \infty} \Phi_n^{-1}(\mathcal{U}) \right) \leq \liminf_{n \rightarrow \infty} \nu_G \circ \Phi_n^{-1}(\mathcal{U}).$$

But, Corollary 14 tells us that  $\nu_G$  is  $\Phi_n$ -invariant. Thus, the above sequence of inequalities tells us that  $\nu_G(\mathcal{E}) \leq \nu_G(\mathcal{U}) < \epsilon$ . Since  $\epsilon$  was arbitrary, we conclude that  $\nu_G(\mathcal{E}) = 0$  as desired.  $\square$

## 7. EXAMPLES OF DEVIOUS COVERS

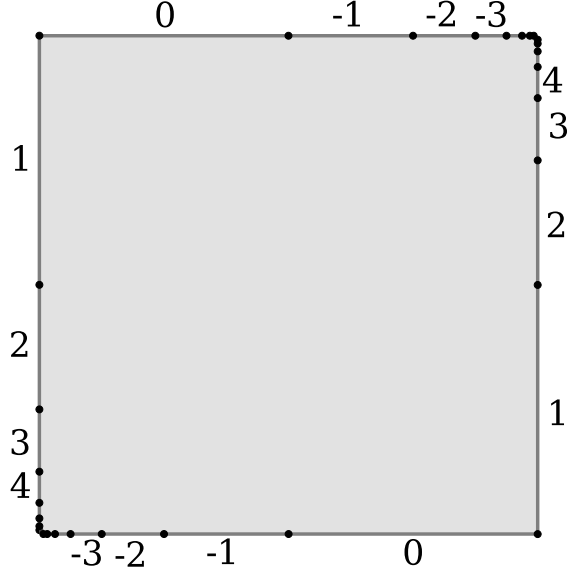


FIGURE 1. Chamanara's surface.

**7.1. Chamanara's surfaces.** We introduce a surface first studied by Chamanara in [Cha04]. (See also the related work [CGL06].) The surface is built from a closed  $1 \times 1$  square with each of the edges subdivided into intervals of length  $\frac{1}{2^k}$  for  $k \in \mathbb{N}$  as indicated in Figure 1. The vertical intervals of equal length are then glued together by translation, and we do the same to the horizontal intervals. The intervals being identified have been labeled by the same integers in the figure. The endpoints of these intervals being glued and the corners of the square are discarded to give the space a translation structure.

Let  $(S_2, \alpha_2)$  denote the surface depicted in Figure 1. We note that the surface has an affine automorphism  $\phi_2$  whose derivative is

$$D(\phi_2) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \in V(S_2, \alpha_2).$$

To see this observe that the image of the square under this linear map is a  $\frac{1}{2} \times 2$  rectangle. If we push this rectangle into the surface allowing the rectangle only to pass through the edge labeled zero in the figure, we see the identifications are respected and thus this map determines an affine automorphism  $\phi_2$  of the surface.

The vertical and horizontal flows on Chamanara's surface  $(S_2, \alpha_2)$  can be seen as the suspension flow over the dyadic odometer. To see this, it suffices to consider the dyadic odometer as an infinite interval exchange map on  $[0, 1]$  defined for  $x \in [0, 1]$ , as the the dyadic odometer on the dyadic expansion  $(x_1, x_2, \dots) \in X_2$  of  $x = \sum x_i 2^{-i}$ . As such, the map induced on the cross section  $C$  of the horizontal flow consisting of the "left side" of Chamanara's surface, as depicted in Figure 1, is isomorphic to the dyadic odometer.

More generally, for integers  $n \geq 2$ , one can construct a homeomorphic translation surface  $(S_n, \alpha_n)$  by letting the identified sides in Figure 1 be of length  $\frac{n-1}{n^k}$  for  $k \in \mathbb{N}$ . This surface admits an affine automorphism with diagonal derivative and eigenvalues of  $n$  and  $\frac{1}{n}$ . As in the case of the dyadic odometer, this surface admits a section which is the  $n$ -adic odometer.

**Remark 20** (Veech groups). *The Veech group  $V(S_n, \alpha_n)$  is known to be generated by two parabolics. See [Cha04, Theorem B].*

We now give a proof of Theorem 7 as a straight forward consequence of Corollary 6.

*Proof of Theorem 7.* Fix an integer  $n \geq 2$ . Recall that the  $n$ -adic odometer is uniquely ergodic, because it can be understood as a minimal rotation of a compact abelian group. Therefore, the translation flow on the surface  $(S_n, \alpha_n)$  is also uniquely ergodic. Observe that the Teichmüller flow is periodic, because the affine automorphism  $\phi_n$  has diagonal derivative. If  $d \geq 2$  and  $G \subset \Pi_d$  is a subgroup acting transitively on  $\{1, \dots, d\}$ , then corollaries 3 and 6 imply that  $m_G$ -almost every cover has a uniquely ergodic translation flow, where  $m_G$  is the probability measure from §4.2. Taking a random cover is equivalent to taking  $d$  copies of the unit square and gluing the edges making up  $(S_n, \alpha_n)$  according to an element of  $G$  taken at random. The translation flow on a cover  $(\tilde{S}_n, \tilde{\alpha}_n)$  is canonically the suspension flow of a skew product over the  $n$ -adic odometer, thus the measure  $m_G$  induces a measure on the space of such skew products. We observe that this measure is exactly given by the measure  $\mu_{n,G}$  from the statement of the theorem, and thus  $\mu_{n,G}$ -almost every skew product is uniquely ergodic.  $\square$

For the remainder of the section, we will concentrate on the simplest of Chamanara's surfaces,  $(S_2, \alpha_2)$ , which will denote by  $(S, \alpha)$ . We will simplify the notation for the pseudo-Anosov  $\phi_2$  by denoting it by  $\phi$ .

Our goal with the remainder of this section is to investigate what happens when we have a connected cover  $(\tilde{S}, \tilde{\alpha})$  which when iterated by application of  $\phi$  only accumulates on disconnected covers. It can be observed that the connectivity of  $(\tilde{S}, \tilde{\alpha})$  does not guarantee the ergodicity of the (horizontal) translation flow, since the connectivity can be arranged with only the gluings of horizontal edges when building the cover as  $d$  copies of the unit square with edge identifications. We will give a proof of the fact that such connected devious covers are dense inside of the space of covers.

It is interesting to consider whether there are connected covers which accumulate only on disconnected covers under the Teichmüller deformation but whose translation flow is nonetheless uniquely ergodic. We show such covers exist. This is analogous to slow divergence giving rise to unique ergodicity in the classical setting of closed translation surfaces.

We will now introduce some more notation and formally state the results mentioned in the previous two paragraphs. Let  $G$  be a subgroup of the symmetric group  $\Pi_d$  with  $d \geq 2$  which acts transitively on  $\{1, \dots, d\}$ . Recall that the space of covers of  $(S, \alpha)$  with monodromy in  $G$  can be thought of as

$$\text{Cov}_G(S, \alpha) = \Pi_d \backslash \text{Hom}(\pi_1(S, s_0), G).$$

(Note that  $\ker D$  is trivial; the surface has no translation automorphisms.) If  $h$  is a homomorphism from  $\pi_1(S, s_0)$  to  $G$ , we use  $[h]$  to denote its equivalence class in  $\text{Cov}_G(S, \alpha)$ . The disconnected covers in  $\text{Cov}_G(S, \alpha)$  are given by

$$\bigcup_{H \in \mathcal{H}} \Pi_d \backslash \text{Hom}(\pi_1(S, s_0), H),$$

where  $\mathcal{H}$  denotes the collection of all subgroups of  $G$  which fail to act transitively on  $\{1, \dots, d\}$ .

Recall that formally, the affine automorphism  $\phi^{-1}$  does not act on the fundamental group of  $S$ . We choose a basepoint  $s_0$  in the interior of the square near the southwest corner of the square in Figure 1. To get an action on the fundamental group, we need to select a curve joining  $s_0$  to  $\phi^{-1}(s_0)$  as described by equation 5. Because we chose  $s_0$  in the interior of the square near the southwest corner, its image  $\phi^{-1}(s_0)$  will also lie near the southwest corner and in the interior of the square. We specify  $\delta$  to be a curve joining  $s_0$  to  $\phi^{-1}(s_0)$  while not leaving the interior of the square. Then (as in equation 5), we define

$$\phi_\delta^{-1} : \pi_1(S, s_0) \rightarrow \pi_1(S, \phi^{-1}(s_0)); \quad [\beta] \mapsto [\delta \bullet (\phi^{-1} \circ \beta) \bullet \delta^{-1}].$$

**Theorem 21** (Non-ergodic covers). *Let  $h \in \text{Hom}(\pi_1(S, s_0), G)$ . The translation flow on the cover  $(\tilde{S}_h, \tilde{\alpha}_h)$  associated to  $[h]$  is non-ergodic whenever there is an  $H \in \mathcal{H}$  so that every accumulation point of  $h \circ \phi_\delta^{-n}$  (as  $n \rightarrow \infty$ ) lies in  $\text{Hom}(\pi_1(S, s_0), H)$ . For any  $H \in \mathcal{H}$ , there exists a dense collection of connected covers so that every accumulation point lies in  $\Pi_d \setminus \text{Hom}(\pi_1(S, s_0), H)$  but no accumulation point lies in  $\Pi_d \setminus \text{Hom}(\pi_1(S, s_0), H')$  for any proper subgroup  $H' \subset H$ .*

**Theorem 22.** *Suppose  $G$  is a subgroup of  $\Pi_d$ , and  $H_1, H_2 \subset G$  are subgroups which do not act transitively on  $\{1, \dots, d\}$ , but the group generated by the elements of  $H_1 \cup H_2$  does act transitively. Then, there are finite covers of  $(S, \alpha)$  with monodromy in  $G$  whose translation flow is uniquely ergodic, but whose orbit under  $\phi$  accumulates only on surfaces in the collection of disconnected covers,*

$$\Pi_d \setminus \left( \text{Hom}(\pi_1(S, s_0), H_1) \cup \text{Hom}(\pi_1(S, s_0), H_2) \right).$$

**Remark 23.** *It seems likely that there are also non-ergodic covers whose orbits under  $\phi$  accumulate as Theorem 22. We do not investigate this question.*

The key to proving these results is an understanding of the action of  $\phi_\delta^{-1}$  on the fundamental group. In order to describe this action, we select a generating set. For each integer  $n$ , we will let  $\gamma_n \in \pi_1(S, s_0)$  be a homotopy class of curves which start and end at the basepoint. If  $n \leq 0$ , we define  $\gamma_n$  to contain the curves which move downward from the basepoint passing through the horizontal edge labeled  $n$  and returning to the basepoint without passing through any other labeled edges. We similarly define  $\gamma_n$  for  $n > 0$  to contain the curves which move rightward over the vertical edge labeled  $n$ . Observe that  $\pi_1(S, s_0)$  is freely generated by  $\{\gamma_n : n \in \mathbb{Z}\}$ .

We also define  $\phi_\delta$  as the inverse of the map  $\phi_\delta^{-1}$ . The following describes the actions of  $\phi_\delta^{-1}$  and  $\phi_\delta$  on our basis for  $\pi_1(S, s_0)$ .

**Proposition 24.** *For each  $n \in \mathbb{N}$ , we have*

$$\phi_\delta^{-1}(\gamma_n) = \begin{cases} \gamma_{n+1}\gamma_1^{-1} & \text{if } n < 0 \\ \gamma_1 & \text{if } n = 0 \\ \gamma_1\gamma_{n+1} & \text{if } n > 0, \end{cases} \quad \text{and} \quad \phi_\delta(\gamma_n) = \begin{cases} \gamma_{n-1}\gamma_0 & \text{if } n \leq 0 \\ \gamma_0 & \text{if } n = 1 \\ \gamma_0^{-1}\gamma_{n-1} & \text{if } n > 1. \end{cases}$$

This proposition may be proved by inspecting the action of  $\phi_\delta^{-1}$ , and we leave the details to the reader.

In order to work with homomorphisms  $h : \pi_1(S, s_0) \rightarrow G$ , we define the  $G$ -sequence of a homomorphism  $h$  to be the bi-infinite sequence of elements of  $G$  defined by

$$g_m = h \circ \phi_\delta^{-m}(\gamma_1) \quad \text{for } m \in \mathbb{Z}.$$

This sequence encodes  $h$  in a way which is natural with respect to the action of  $\phi$ . The following Lemma implies that the map  $h \mapsto \langle g_m \rangle_{m \in \mathbb{Z}}$  is a homeomorphism. It follows that this map conjugates the action of  $\phi_\delta^{-1}$  on  $\text{Hom}(\pi_1(S, s_0), G)$  to the shift map on  $G^{\mathbb{Z}}$ .

**Lemma 25.** *Let  $\langle g_m \rangle$  be the  $G$ -sequence of a homomorphism  $h : \pi_1(S, s_0) \rightarrow G$ . Then, for each  $k, n \in \mathbb{Z}$ ,*

$$h \circ \phi_\delta^{-k}(\gamma_n) = \begin{cases} g_{k+n-1} g_{k+n}^{-1} g_{k+n+1}^{-1} \cdots g_{k-1}^{-1} & \text{if } n < 0 \\ g_{k-1} & \text{if } n = 0 \\ g_k & \text{if } n = 1 \\ g_k^{-1} g_{k+1}^{-1} \cdots g_{k+n-2}^{-1} g_{k+n-1} & \text{if } n > 1, \end{cases}$$

*Proof.* Observe that by definition  $h \circ \phi_\delta^{-k}(\gamma_1) = g_k$  for all  $k \in \mathbb{Z}$ . This case of  $n = 1$  will serve as a base case for proving the statement holds when  $n \geq 1$ . Note that the formula given in the case of  $n > 1$  can be extended to hold for  $n = 1$  if one allows the (empty) product of inverses  $g_k^{-1} g_{k+1}^{-1} \cdots g_{k+n-2}^{-1}$  to be the identity when  $n = 1$ . So, suppose our formula holds for some  $n \geq 1$  and all  $k$ , we will show it holds for  $n + 1$  and all  $k$ . Using the proposition, we observe that for  $n \geq 1$ ,

$$h \circ \phi_\delta^{-k-1}(\gamma_n) = h \circ \phi_\delta^{-k}(\gamma_1 \gamma_{n+1}) = g_k \cdot h \circ \phi_\delta^{-k}(\gamma_{n+1}).$$

By our inductive hypothesis applied to the left side, we see that

$$g_{k+1}^{-1} g_{k+2}^{-1} \cdots g_{k+n-1}^{-1} g_{k+n} = g_k \cdot h \circ \phi_\delta^{-k}(\gamma_{n+1})$$

which gives us that  $\phi_\delta^{-k}(\gamma_{n+1}) = g_k^{-1} \cdots g_{k+n-1}^{-1} g_{k+n}$ . This proves our formula for  $n + 1$  and all  $k$ . So, by induction, the statement holds for  $n \geq 1$ .

Now we consider the base case of  $n = 0$ . Observe that

$$h \circ \phi_\delta^{-k}(\gamma_0) = h \circ \phi_\delta^{-k+1} \circ \phi_\delta^{-1}(\gamma_0) = h \circ \phi_\delta^{-k+1}(\gamma_1) = g_{k-1}.$$

Again observe that if we treat empty products of negations as the identity that the formula for  $n < 0$  holds for  $n = 0$ . Now suppose the formula holds for some  $n \leq 0$  and all  $k$ . Then, by the proposition and the base case

$$h \circ \phi_\delta^{-k+1}(\gamma_n) = h \circ \phi_\delta^{-k}(\gamma_{n-1} \gamma_0) = h \circ \phi_\delta^{-k}(\gamma_{n-1}) \cdot g_{k-1}.$$

By inductive hypothesis, we see

$$h \circ \phi_\delta^{-k}(\gamma_{n-1}) = h \circ \phi_\delta^{-k+1}(\gamma_n) \cdot g_{k-1}^{-1} = (g_{k+n-2} g_{k+n-1}^{-1} \cdots g_k^{-1}) g_{k-1}^{-1}.$$

This completes the inductive step, proving the statement for all  $n \leq 0$ .  $\square$

Now we prove our theorem on the non-ergodicity of covers.

*Proof of Theorem 21.* Let  $h : \pi_1(S, s_0) \rightarrow G$  be a homomorphism, and suppose that all accumulation points of  $h \circ \phi^{-k}$  as  $k \rightarrow \infty$  lie in  $\text{Hom}(\pi_1(S, s_0), H)$  for some subgroup  $H \subset G$ , where  $H$  does not act transitively on  $\{1, \dots, d\}$ . Then, there is a  $K$  so that for each  $k \geq K$ ,  $h \circ \phi^{-k}(\gamma_1) \in H$ . In terms of the  $G$ -sequence  $\langle g_k \rangle$  of  $h$ , we see that  $g_k \in H$  for  $k \geq K$ . Therefore, by the lemma above,

$$h \circ \phi^{-K}(\gamma_n) = g_K^{-1} g_{K+1}^{-1} \cdots g_{K+n-2}^{-1} g_{K+n-1} \in H$$

for all  $n \geq 1$  (where when  $n = 1$ , the product  $g_K^{-1} g_{K+1}^{-1} \cdots g_{K+n-2}^{-1}$  is taken to be the identity). Observe that the horizontal straight-line flow on the surface  $(S, \alpha)$  only crosses the intervals with positive label in Figure 1. Now consider the cover associated to  $h \circ \phi_\delta^{-K}$ . The cover can

be built from copies of the square indexed by  $\{1, \dots, d\}$  with edges identified according to  $h \circ \phi_\delta^{-K}$ . In particular, the intervals with positive label are glued only according to elements of  $H$ . Thus, points in copy  $i \in \{1, \dots, d\}$  only can reach the copies of the square indexed by elements of the orbit  $H(i)$ , and by assumption  $H(i) \neq \{1, \dots, d\}$ . In particular the union of the squares indexed by  $H(i)$  gives an invariant set with measure strictly between zero and full measure. Note that  $\phi^K$  induces an affine homeomorphism with diagonal derivative from the cover associated to  $h$  to the cover associated to  $h \circ \phi_\delta^{-K}$ , so pulling back this invariant set gives an invariant subset of the straight line flow for the cover associated to  $h$  with intermediate measure.

Finally, we note that the collection of sequences  $\langle g_k \rangle$  so that there is a  $K$  with  $g_k \in H$  for  $k \geq K$  is dense inside  $G^\mathbb{Z}$ . This remains true if we also insist that the subgroup generated by  $\{g_k : k \in \mathbb{Z}\}$  acts transitively on  $\{1, \dots, d\}$ . Any  $h \in \text{Hom}(\pi_1(S, s_0), G)$  whose  $G$ -sequence has these property is associated to a connected cover with non-ergodic translation flow. The map which recovers  $h$  from its  $G$ -sequence is a homeomorphism, so our set of connected but non-ergodic covers is dense. To guarantee that no accumulation point lies in a smaller subgroup, we can further choose  $g_k$  for  $k \geq K$  to periodically list all elements of  $H$  with period  $|H|$ . Elements of this form remain dense, and since the  $G$ -sequence is a homeomorphism any limiting homeomorphism would have to have a  $G$ -sequence which periodically lists the elements of  $H$ . Thus, no such limiting homeomorphism could lie in  $\text{Hom}(\pi_1(S, s_0), H')$  for a proper subgroup  $H' \subset H$ .  $\square$

Now we will move toward proving our statement about the existence of covers with ergodic translation flow whose  $\phi$ -orbits accumulate only on disconnected covers. Recall that our theorem dealt with the situation when we have two subgroups  $H_1$  and  $H_2$  of  $G$  which do not act transitively on  $\{1, \dots, d\}$  but which together act transitively.

Let  $a \in \mathbb{Z}$  and  $A = \{n \in \mathbb{Z} : n \geq a\}$ . Let  $p : A \rightarrow \{0, 1, 2\}$  be an arbitrary function with the property that  $p(n) + p(n+1) \neq 3$  for each  $n \in A$ . For such a map  $p$  and an  $n \in A$ , we define the *preimage interval* containing  $n$ ,  $I(p, n) \subset \mathbb{Z}$ , to be the maximal collection of consecutive elements of  $A$  containing  $n$  so that  $n \in I(p, n)$  and  $p$  is constant on  $I(p, n)$ . We'll say that a sequence  $\langle g_n \rangle \in G^\mathbb{Z}$  is *p-ready* if the following two statements hold for each  $n \in A$ :

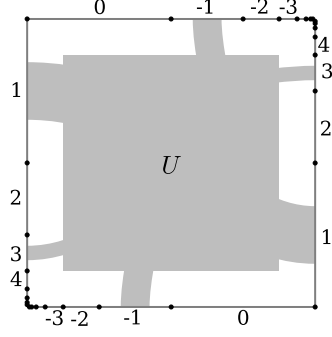
- If  $p(n) = 0$ , then  $g_n \in H_1 \cap H_2$ .
- If  $p(n) \in \{1, 2\}$ , then  $g_n \in H_{p(n)}$  and  $\{g_k : k \in I(p, n)\}$  generates  $H_{p(n)}$ .

Let  $|I(p, n)|$  denote the number of integers in  $I(p, n)$ .

**Proposition 26** (Criterion for disconnected accumulation points). *Suppose that  $h$  has  $G$ -sequence  $\langle g_n \rangle$ , and that this sequence is  $p$ -ready. If the lengths of preimage intervals on which  $p$  is zero tend to infinity, then every accumulation point of  $h \circ \phi_\delta^{-n}$  as  $n \rightarrow \infty$  corresponds to a disconnected cover. These accumulation points all lie in  $\text{Hom}(\pi_1(S, s_0), H_1)$  or  $\text{Hom}(\pi_1(S, s_0), H_2)$ .*

*Proof.* Let  $h \in \text{Hom}(\pi_1(S, s_0), G)$  and let  $\langle g_n \rangle$  be its  $G$ -sequence. Suppose there is a subsequence  $h \circ \phi^{-k_j}$  which converges to  $h_\infty$  representing a connected cover. Let  $\langle g_n^\infty \rangle$  be the  $G$ -sequence of  $h_\infty$ . Observe that the  $G$ -sequence of  $h \circ \phi_\delta^{-k}$  is given by  $\langle g_n^k = g_{n+k} \rangle$ . So, for each  $n$ , there is a  $J$  so that  $g_{n+k_j} = g_n^\infty$  for  $j > J$ . Furthermore, the collection  $\{g_n^\infty\}$  generates a subgroup of  $G$  which acts transitively on  $\{1, \dots, d\}$ . Thus, we can find integers  $b < c$  so that  $\{g_n^\infty : b \leq n \leq c\}$  generates a subgroup of  $G$  which acts transitively. But then for sufficiently large  $j$ , we see that  $\{g_{n+k_j} : b \leq n \leq c\}$  generates a transitive subgroup.



FIGURE 2. A subsurface  $U$  when  $J = \{-1, 1, 3\}$ .

Let  $I_j = \{b + k_j, \dots, c + k_j\}$ . Observe that since  $\langle g_n^\infty \rangle$  is  $p$ -ready, it must be true that for such large  $j$  with  $I_j \subset A$ , we have  $\{1, 2\} \subset p(I_j)$ . But since  $p$  takes both the values of 1 and 2 on  $I_j$ , it must also take the value zero (otherwise  $p(n) + p(n+1) = 3$  for some  $n$ ). Moreover, this  $m \in I_j$  so that  $p(m) = 0$  must separate occurrences of 1 from the occurrences of 2. Thus, we see  $I(p, m) \subset I_j$ . But then we have found  $m$  arbitrarily large with  $p(m) = 0$  and  $|I(p, m)|$  less than the constant  $c - b + 1 = |I_j|$ . In other words, the preimage intervals on which  $p$  is zero do not tend in length to infinity.  $\square$

Our proof will combine the above with the following lemma, which will allow us to apply the ergodicity criterion of given in Theorem 18.

**Lemma 27.** *Let  $G \subset \Pi_d$  with  $d \geq 2$  be a subgroup which acts transitively on  $\{1, \dots, d\}$ . For each finite subset  $J \subset \mathbb{Z}$  and for each  $\eta > 0$ , there is a constant  $c = c(\eta, J) > 0$  so that for each  $h \in \text{Hom}(\pi_1(S, s_0), G)$  that satisfies the condition that the subgroup generated by  $\{h(\gamma_j) : j \in J\}$  acts transitively on  $\{1, \dots, d\}$ , there is a subsurface  $\tilde{U}$  of the cover  $(\tilde{S}_h, \tilde{\alpha}_h)$  associated to  $h$ , so that  $\text{Area}(\tilde{S}_h \setminus \tilde{U}) < \eta$  and*

$$\epsilon(t)^4 \mathcal{D}_t^{-2} > c \quad \text{for each } t \in \mathbb{R} \text{ with } |t| \leq \frac{\ln(2)}{2},$$

where  $\epsilon(t)$  represents a lower bound for the distance from points in  $g^t(\tilde{U})$  to points in the completion of  $g^t(\tilde{S}_h, \tilde{\alpha}_h)$ , and  $\mathcal{D}_t$  is an upper bound for the diameter of  $g^t(\tilde{U})$ .

We note that the expression  $\epsilon(t)^4 \mathcal{D}_t^{-2}$  is the function inside the integral in Theorem 18 in the special case of considering a single connected subsurface (i.e.,  $C_t \equiv 1$ ).

*Proof.* We will explicitly describe how to build  $\tilde{U}_h \subset (\tilde{S}_h, \tilde{\alpha}_h)$ . We will define  $U$  to be a subsurface of  $(S, \alpha)$ , which only depends on  $J$  and  $\eta$ . Recall that  $(S, \alpha)$  was built from a single square with edge identifications. To build  $U$ , start with a smaller concentric square with area equal to  $1 - \frac{\eta}{2d}$ . Then for each  $j \in J$ , consider the edge of the surface labeled  $j$ . Attach to the concentric square a “handle” passing through the edge, which stays a bounded distance away from  $\Sigma$  (the points added in the metric completion). See Figure 2 for an example. Then, we define  $\tilde{U}_h$  to be the preimage of  $U$  under the covering map  $\tilde{S}_h \rightarrow S$ . We note that  $\tilde{U}_h$  is connected because of our condition that  $\{h(\gamma_j) : j \in J\}$  generates  $G$ .

Note that the minimal distance from the boundary of  $g^t(\tilde{U})$  to the metric completion in  $g^t(\tilde{S}_h, \tilde{\alpha}_h)$  is the same as the minimal distance from  $g^t(U)$  to the metric completion in  $g^t(S, \alpha)$ . This distance is always positive and varies continuously in  $t$ , so we can get a uniform lower

bound which holds when  $|t| \leq \frac{1}{2} \ln 2$ . Similarly, the diameter of  $g^t(\tilde{U}_h)$  varies continuously in  $t$ , so we can get an upper bound on the diameter which is uniform in  $t$  with  $|t| \leq \frac{1}{2} \ln 2$ . Finally observe that up to translation equivalence there are only finitely many  $\tilde{U}_h$  as we vary  $h$ . (The geometry of  $\tilde{U}_h$  only depends on the restriction of  $h$  to  $J$ .) Thus, we can get an upper bound on the diameter of which is uniform in both  $h$  satisfying our condition and  $t$  satisfying  $|t| \leq \frac{1}{2} \ln 2$ .  $\square$

We need a mechanism to ensure that  $h$  satisfies the condition of the lemma (i.e., for some fixed  $J \subset \mathbb{Z}$  the subgroup generated by  $\{h(\gamma_j) : j \in J\}$  acts transitively on  $\{1, \dots, d\}$ ), given conditions on the  $G$ -sequence  $\langle g_n \rangle$  of  $h$ . This mechanism is a corollary of Lemma 25.

**Corollary 28.** *Suppose  $h$  has  $G$ -sequence  $\langle g_n \rangle$ . Let  $m, k \in \mathbb{Z}$  with  $m > 0$ . Then, if the subgroup generated by  $\{g_j : k - m \leq j \leq k + m\}$  acts transitively on  $\{1, \dots, d\}$ , then so does the subgroup generated by  $\{h \circ \phi_\delta^{-k}(\gamma_j) : -m + 1 \leq j \leq m + 1\}$ .*

*Proof.* Using Lemma 25, we can find an expression for each  $g_j$  with  $k - m \leq j \leq k + m$  in terms of a  $\{h \circ \phi_\delta^{-k}(\gamma_j) : -m + 1 \leq j \leq m + 1\}$ .  $\square$

*Proof of Theorem 22.* Let  $a \in \mathbb{Z}$  and  $A = \{n \in \mathbb{Z} : a \leq n\}$ , and let  $p : A \rightarrow \{0, 1, 2\}$  be a function so that  $p(n) + p(n+1) \neq 3$  for all  $n \in A$  as above. We call  $p$  *alternating* if whenever  $n \in A$  and  $p(n) = 0$ , we have  $p(b-1) \neq p(c+1)$ , where  $b$  and  $c$  are the endpoints of the preimage interval  $I(p, n) = \{b, b+1, \dots, c\}$ .

For each positive integer  $m$ , let  $J_m = \{n \in \mathbb{Z} : |n| \leq m\}$ . Let  $\text{Hom}_m \subset \text{Hom}(\pi_1(S, s_0), G)$  be the collection of all  $h$  so that the subgroup generated by  $\{h(\gamma_n) : n \in J_m\}$  acts transitively on  $\{1, \dots, d\}$ . Let  $\langle \eta_i \rangle_{i>0}$  be a sequence of positive reals tending to zero monotonically as  $i \rightarrow \infty$ . Then applying the prior lemma yields a constant  $c_{i,m} = c(\eta_i, J_m) > 0$  so that  $h \in \text{Hom}_m$  implies that for  $\eta = \eta_i$ , the function being integrated in Theorem 18 is larger than  $c_{i,m}$  on the interval  $[-\frac{1}{2} \ln 2, \frac{1}{2} \ln 2]$ .

For each  $i > 0$  and  $m > 0$ , choose an integer  $N_{m,i} \geq 0$  so that for each  $i$ ,

$$\sum_{m=1}^{\infty} N_{m,i} c_{i,m} = +\infty.$$

This choice has the consequence that if  $h \in \text{Hom}(\pi_1(S, s_0), G)$  and  $m_*$  is some constant, and there is a pairwise disjoint collection of subsets of positive integers  $K_m$  for integers  $m \geq m_*$ , so that for each  $k \in K_m$ ,  $h \circ \phi^{-k} \in \text{Hom}_m$ , then the integral in Theorem 18 in the case of  $\eta = \eta_i$  is infinite. Now define a new sequence of integers by

$$M_m = \max \{N_{m,1}, N_{m,2}, \dots, N_{m,m}\}.$$

Observe that if  $h$  is given and for  $m \geq m_*$ , where  $m_*$  is some constant, there is a pairwise disjoint collection of subsets of positive integers

$$\{K_m \subset \mathbb{N} : m \geq 1\} \quad \text{satisfying} \quad |K_m| \geq M_m$$

so that  $k \in K_m$  implies  $h \circ \phi^{-k} \in \text{Hom}_m$ , then the integral in Theorem 18 in the case of  $\eta = \eta_i$  is infinite for every  $i$ . (The contribution to the integral by values of  $t$  with  $t \in [(k - \frac{1}{2}) \ln 2, (k + \frac{1}{2}) \ln 2]$  is at least  $c_{i,m} \ln 2$ .) It follows that the integral is infinite for every  $\eta > 0$ , because if the integral is infinite for  $\eta_i$ , then it is infinite for any larger value  $\eta$ , which can be seen by using the same subsurfaces.

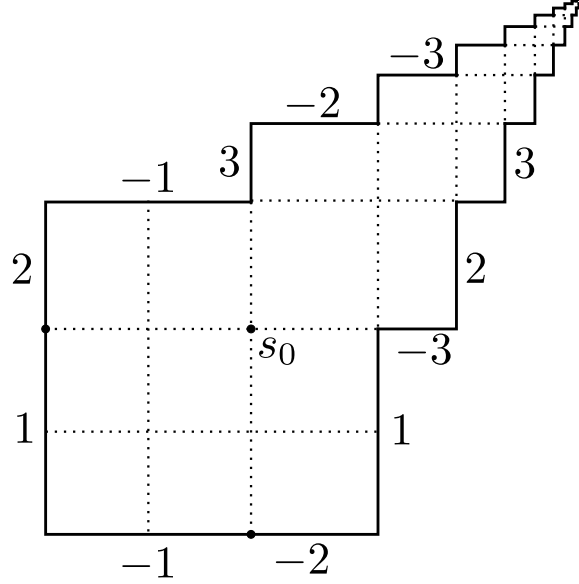


FIGURE 3. The ladder surface.

Now we will produce a cover satisfying the theorem. For  $i \in \{1, 2\}$ , let  $L_i$  be the minimal number of elements of  $H_i$  needed to generate. Choose a positive integer  $m_*$  so that  $2m_* + 1 \geq L_1 + L_2 + 1$ . Now choose a sequence of integers  $\{\ell_i \geq 0\}$  so that  $\liminf_{i \rightarrow \infty} \ell_i = \infty$  and so that for each  $m \geq m_*$ , there are at least  $M_m$  indices  $i$  satisfying  $L_1 + L_2 + \ell_i = 2m + 1$ . Using the sequence  $\{\ell_i\}$ , we define  $p : \mathbb{N} \rightarrow \{0, 1, 2\}$ , which we interpret as a infinite word  $p(1)p(2)p(3) \dots$  in the alphabet  $\{0, 1, 2\}$  to be

$$p = 1^{L_1} 0^{\ell_1} 2^{L_2} 0^{\ell_2} 1^{L_1} 0^{\ell_3} 2^{L_2} 0^{\ell_4} \dots$$

(Here, exponentiation denotes repetition.) Then we define a sequence  $\{g_n\} \in G^{\mathbb{Z}}$  which is  $p$ -ready as defined on page 24. We observe that whenever  $L_1 + L_2 + \ell_i = 2m + 1$ , the adjacent collection of symbols of the form  $1^{L_1} 0^{\ell_i} 2^{L_2}$  or  $2^{L_2} 0^{\ell_i} 1^{L_1}$  has a center index  $k$  for which  $h \circ \phi^{-k} \in \text{Hom}_m$ , where  $h \in \text{Hom}(\pi_1(S, s_0), G)$  is selected so that its  $G$ -sequence is  $\{g_n\}$ . Then the conclusions of the prior paragraph apply for the cover  $(\tilde{S}_h \tilde{\alpha}_h)$ , i.e., the translation flow is ergodic and hence uniquely ergodic by Corollary 3. On the other hand, because  $\liminf_{i \rightarrow \infty} \ell_i = \infty$ , we satisfy Proposition 26 and all limiting surfaces can be constructed by monodromy homomorphisms in  $\text{Hom}(\pi_1(S, s_0), H_1)$  or  $\text{Hom}(\pi_1(S, s_0), H_2)$ , both of which parameterize disconnected covers.  $\square$

**7.2. The ladder surface.** For this subsection, we let  $(S, \alpha)$  denote the infinite genus translation surface of Figure 3, which we call the *ladder surface*. (The surface is built using Thurston's construction from a graph resembling a ladder with a stand.) It can be constructed from a region in the plane bounded by countably many horizontal and vertical edges. We have labeled the edges using the set  $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ . Edges with the same label are glued by translation to form the surface. Let  $\varphi$  denote the golden ratio,  $\frac{\sqrt{5}+1}{2}$ . An edge with label  $j \in \mathbb{Z}^*$  has length given by  $\varphi^{-|j|}$ . This information specifies the geometry of the surface. We have also selected a basepoint  $s_0$  for our surface in the figure.

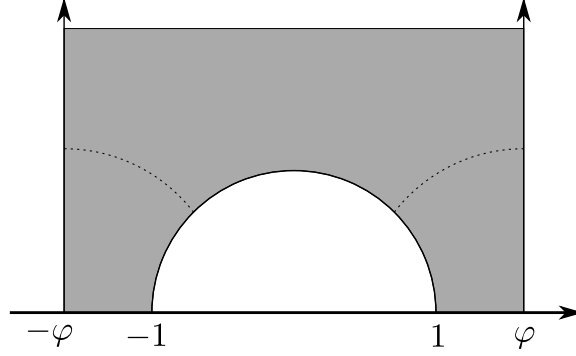


FIGURE 4. A fundamental domain for the action of  $V'$  on the upper half plane. The geodesic between  $-1$  and  $1$  is preserved by the reflection  $\rho$ , and  $\psi$  acts by translation by  $2\varphi$ .

The surface  $(S, \alpha)$  admits two affine automorphisms  $\psi$  and  $\rho$  whose derivatives are given by

$$D(\psi) = \begin{bmatrix} 1 & 0 \\ 2\varphi & 1 \end{bmatrix} \quad \text{and} \quad D(\rho) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The first automorphism,  $\psi$ , performs a single left Dehn twist in each of countably many vertical cylinders on  $(S, \alpha)$ . These horizontal cylinders are separated by dotted horizontal lines in Figure 3. The automorphism  $\rho$  performs a Euclidean reflection in the line of symmetry of the figure of slope 1. There is another multi-twist then given by  $\rho \circ \psi \circ \rho$ , which preserves the horizontal cylinders depicted in the figure. All these affine automorphisms fix our basepoint  $s_0$ . Let  $Aff'$  be the known affine automorphism group,  $\langle \psi, \rho \rangle$ , which is isomorphic to the free product  $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})$ .

The group  $SL(2, \mathbb{R})$  can be identified with the unit tangent bundle of the hyperbolic plane, where the hyperbolic plane is  $O(2) \backslash SL_{\pm}(2, \mathbb{R})$ , where  $SL_{\pm}(2, \mathbb{R})$  is the group of matrices of determinant  $\pm 1$ . The group  $SL_{\pm}(2, \mathbb{R})$  then acts on the right by isometry. The unit speed geodesics have the form  $t \mapsto g^t A$  for some  $A \in SL_{\pm}(2, \mathbb{R})$ , where  $g^t(x, y) = (e^{-t}x, e^ty)$  as in the introduction. The boundary of the hyperbolic plane can be identified with  $\mathbb{RP}^1 = \mathbb{R}^2/(\mathbb{R} \setminus \{0\})$ . The geodesic  $t \mapsto g^t A$  converges as  $t \rightarrow +\infty$  to the boundary point representing the vectors which are contracted to zero as  $t \rightarrow \infty$ , the projective equivalence class of

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We identify the hyperbolic plane  $O(2) \backslash SL_{\pm}(2, \mathbb{R})$  with the upper half plane in such a way so that the projective class of  $(x, y)$  viewed as a boundary point is identified with the slope  $y/x \in \mathbb{R} \cup \{\infty\}$ .

We let  $V'$  denote the group of derivatives of  $Aff'$ . This group acts discretely on the hyperbolic plane, and Figure 4 shows a fundamental domain for the action. A geodesic  $t \mapsto g^t AV'$  in  $SL_{\pm}(2, \mathbb{R})/V'$  is non-divergent if and only if the endpoint of the geodesic in the hyperbolic plane lies in the horospherical limit set, which in this case is the limit set of  $V'$  with fixed points of parabolics removed [BM74, Theorem 2]. The limit set in this case is a Cantor set of Hausdorff dimension larger than  $\frac{1}{2}$  [Hoo10, Remark 4.1].

**Theorem 29.** *Let  $A \in SL_{\pm}(2, \mathbb{R})$ , and  $(S, \alpha)$  be the ladder surface.*

- If the endpoint of the geodesic  $t \mapsto g^t A$  is in the horocyclic limit set of  $V'_+$ , then the translation flow on  $A(S, \alpha)$  is ergodic.
- If in addition, the endpoint of this geodesic is not fixed by a conjugate of the commutator,  $D([\rho, \psi])$ , in  $V'_+$ , then the translation flow on  $A(S, \alpha)$  is uniquely ergodic.

*Proof.* The first statement follows from the remarks above and Theorem 18 (originally in [Tre14]). The second statement is an application of Theorem H.5 of [Hoo10].  $\square$

**Remark 30** (Boundary of convex core). *The geodesic in the surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$  with monodromy given by  $D([\rho, \psi])$  is the boundary of the convex core of this surface. (The convex core contains all geodesics which are non-divergent in both forward and backward time.) This geodesic is depicted using dotted lines in Figure 4.*

We will be considering double covers of the ladder surface. For  $i \in \mathbb{Z}^*$ , we let  $\gamma_i$  be one of the two homotopy class of loops which start at the basepoint, cross only the edge labeled  $i$ , and return to the basepoint. These two homotopy classes are inverses in  $\Gamma = \pi_1(S, s_0)$ . We make the choice of  $\gamma_i$  so  $\gamma_i$  moves rightward across the vertical edge labeled  $i$  if  $i > 0$ , and moves upward over the horizontal edge labeled  $i$  if  $i < 0$ . These curves freely generate the fundamental group, which we denote by  $\Gamma = \pi_1(S, s_0)$ .

Let  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ . The space of double covers  $\text{Cov}_2(S, \alpha)$  is precisely  $\text{Hom}(\Gamma, \mathbb{Z}_2)$ , that is  $\text{Cov}_2(S, \alpha) = \text{TopCov}_2(S, \alpha)$ , since the surface admits no translation automorphisms and  $\mathbb{Z}_2$  acts trivially on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  by conjugation. Since our basepoint is fixed by the affine automorphisms in  $\text{Aff}'$ , we have a canonical action of each element in  $\text{Aff}'$  on the space of covers.

**Notation 31.** Let  $h \in \text{Hom}(\Gamma, \mathbb{Z}_2)$ . We will write  $h(i)$  to abbreviate  $h(\gamma_i)$  for  $i \in \mathbb{Z}^*$ . (The space of covers  $\text{Cov}_2(S, \alpha)$  is identified with  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  which in turn can be identified with the set of all functions  $\mathbb{Z}^* \rightarrow \mathbb{Z}_2$ .)

It can be observed that the generators of  $\text{Aff}'$  act on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  as follows:

$$(16) \quad (\psi_*(h))(i) = \begin{cases} h(i) & \text{if } i < 0, \\ h(i) + h(-2) & \text{if } i = 1, \\ h(i) + h(-2) + h(-3) & \text{if } i = 2, \\ h(i) + h(-i-1) + h(-i) + h(-i+1) & \text{if } i > 2. \end{cases}$$

$$(17) \quad (\rho_*(h))(i) = h(-i).$$

A key observation is the following:

**Proposition 32.** *The square  $\psi_*^2$  acts trivially on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$ .*

*Proof.* The action of  $\psi_*^2$  preserves the value of  $h(i)$  for  $i < 0$ , and adds a sum of values of  $h$  evaluated at positive integers to the values of  $h(i)$  for  $i > 0$ . Since adding the same number twice is the same as adding zero in  $\mathbb{Z}_2$ ,  $\psi_*^2$  acts trivially.  $\square$

Consider a double cover  $(\tilde{S}_h, \tilde{\alpha}_h) \in \text{Cov}_2(S, \alpha)$  determined by an  $h \in \text{Hom}(\Gamma, \mathbb{Z}_2)$ . Because of the result above, the subgroup

$$\tilde{V}' = \langle R D(\psi_*^2) R^{-1} \mid R \in V' \rangle$$

consists of derivatives of affine automorphisms of  $(\tilde{S}_h, \tilde{\alpha}_h)$  regardless of the choice of  $h$ . This subgroup is clearly normal; let  $\Delta = V'/\tilde{V}'$  be the quotient. This group has a right action as the Deck group of the (orbifold) covering map

$$(18) \quad O(2) \backslash SL_{\pm}(2, \mathbb{R}) / \tilde{V}' \rightarrow O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'.$$

We think of  $SL_{\pm}(2, \mathbb{R}) / \tilde{V}'_+$  as parameterizing possible  $SL_{\pm}(2, \mathbb{R})$ -deformations of a double cover.

The quotient group  $\Delta$  is readily observed to be isomorphic to the infinite dihedral group which we think of as  $Isom(\mathbb{Z})$ , where  $\mathbb{Z}$  is given the standard metric. We can take the associated homomorphism  $\delta : V' \mapsto Isom(\mathbb{Z})$  to be given by

$$(19) \quad \delta(\psi) : n \mapsto -n \quad \text{and} \quad \delta(\rho) : n \mapsto 1 - n.$$

The action of  $V'$  on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  then induces an action of  $Isom(\mathbb{Z})$ , where

$$\delta(\psi)_* = \psi_* \quad \text{and} \quad \delta(\rho)_* = \rho_*.$$

We will conclude the discussion above by giving a cartoon description of the setup above. We can deform our fundamental domain for the  $V'$  action so that it looks like the left side of Figure 5. The advantage here is that two copies of this domain can be joined together by a Euclidean rotation by  $180^\circ$  about the point  $\infty$ . This rotation is then representing the order two action on  $SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$  given by the right action of  $D(\psi)$ . The surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$  can be tiled by copies of this fundamental domain as shown on the right side of the figure. We have drawn this in such a way so that the action of the Deck group  $Isom(\mathbb{Z})$  appears natural. In particular, for each  $n \in \mathbb{Z}$ , we can join together two copies of the fundamental domain together about the point labeled  $\infty_n$ . We call this region  $F_n$ ; these regions are depicted in the figure. The isometry  $\delta(\psi) : n \mapsto -n$  acts by a rotation by  $180^\circ$  about the point  $\infty_0$ . The isometry  $\delta(\rho) : n \mapsto 1 - n$  acts by a Euclidean reflection in line forming the boundary between  $F_0$  and  $F_1$ . We observe that  $Isom(\mathbb{Z})$  in a natural way on these regions. Namely, we can think of these regions as subsets of  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$ , and if  $R \in V'$ , and  $j = \delta(R) \in Isom(\mathbb{Z})$  then

$$F_n R^{-1} = F_{j(n)}.$$

The actual action is somewhat subtle: When  $m \in \mathbb{Z}$  is even, the translation  $\tau^m = \delta((\rho \circ \psi)^m) : n \mapsto n + m$  acts by translation by  $m$  in the figure, while when  $m$  is odd,  $\tau^m$  acts as a glide reflection carrying each  $F_n$  to  $F_{n+m}$ . (When  $m$  is odd, this action of  $\tau^m$  is orientation reversing.)

Now consider a geodesic  $g^t A \tilde{V}'$  in the quotient surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$ . This geodesic descends to a geodesic  $g^t A V'$  in the base surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$ , and in the cases we care about we may assume that this geodesic is recurrent here.

**Proposition 33** (Coding walk). *Let  $A \in SL(2, \mathbb{R})$ . If the geodesic  $g^t A V'$  is non-divergent in forward time on the orbifold  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$ , then there is a sequence of countably many times*

$$0 = t_0 < t_1 < t_2 < \dots$$

*with  $\lim_{k \rightarrow \infty} t_k = +\infty$  and a sequence of integers  $\{n_k : k = 0, 1, 2, \dots\}$  so that  $g^t A \tilde{V}' \in F_{n_k}$  whenever  $t_k < t < t_{k+1}$ . Furthermore,  $n_{k+1} = n_k \pm 1$  for each  $k \geq 0$ .*

In other words, the lift of a recurrent geodesic gives rise to a walk on the integers. We'll call this walk the *coding walk* of the geodesic  $g^t A \tilde{V}'$ .

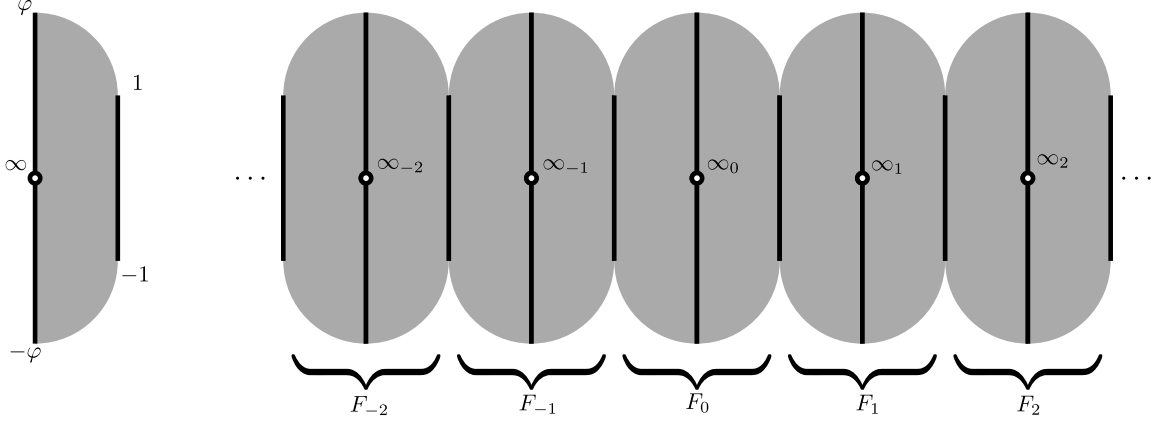


FIGURE 5. *Left:* A caricature of the fundamental domain for the action of  $V'$  on the hyperbolic plane shown in Figure 4. Geodesic boundaries are drawn in black. *Right:* The quotient surface  $O(2)\backslash SL_{\pm}(2, \mathbb{R})/\tilde{V}'$  tiled by copies of this fundamental domain.

*Proof.* If this is not true, then the geodesic change regions only finitely many times. Then the geodesic eventually stays in one region, say  $F_n$ . But, we can see that the only geodesics that stay within  $F_n$  forever are geodesics which exit the cusp  $\infty_n$ , or which limit to a boundary point. (The region  $F_n$  is an annulus with parabolic monodromy.)  $\square$

The coding walk turns out to be useful in characterizing the ergodic properties of the straight-line flow on covers of the ladder surface  $(S, \alpha)$ . We now state our main result.

**Theorem 34** (Main result on double covers of the ladder surface). *Let  $A \in SL(2, \mathbb{R})$  and suppose that the geodesic  $g^t A V'$  is non-divergent in  $SL_{\pm}(2, \mathbb{R})/V'$ . Let  $\langle n_k \rangle$  be the coding walk of the geodesic  $g^t A \tilde{V}'$ . Then:*

- (a) *If the walk recurs, i.e., if there is an  $N \in \mathbb{Z}$  so that the set of visits of the walk to  $\pm N$ ,  $\{k : n_k = N\}$ , is infinite, then the translation flow is uniquely ergodic on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  for any connected double cover  $(\tilde{S}_h, \tilde{\alpha}_h)$  of the ladder surface.*

*If our walk diverges (does not satisfy (a)), then  $\lim_{k \rightarrow \infty} n_k = s\infty$  where  $s \in \{\pm 1\}$  is a sign. We define the growth exponent of the visit count to be*

$$v = \limsup_{N \rightarrow s\infty} \left( \#\{k : n_k = N\} \right)^{\frac{1}{|N|}}.$$

*Then the following hold:*

- (b) *If  $v > \varphi^2$ , then the translation flow is uniquely ergodic on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  for any connected double cover  $(\tilde{S}_h, \tilde{\alpha}_h)$  of the ladder surface.*
- (c) *If  $v < \varphi^2$ , then for any  $h \in \text{Hom}(\Gamma, \mathbb{Z}_2)$ , the translation flow is non-ergodic on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  if and only if  $\lim_{m \rightarrow s\infty} \tau_*^m(h) = \mathbf{0}$ , where  $\tau_*^m = (\rho_* \circ \psi_*)^m$  denotes the action of translation by  $m$  on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  and  $\mathbf{0} \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  is the trivial homomorphism.*

**Remark 35.** *In statement (c), ergodicity implies unique ergodicity unless the geodesic  $t \mapsto g^t A V'$  is the geodesic boundary of the convex core of  $O(2)\backslash SL_{\pm}(2, \mathbb{R})/V'$ . This is an application of lifting unique ergodicity as in Corollary 3 applied to the results in Theorem 29.*

We will prove this theorem in stages below.

*Proof of statement (a) of Theorem 34.* We first observe that the conditions of statement (a) do not allow the geodesic  $g^t AV'$  to be forward asymptotic to the lifted boundary of the convex core of  $O(2) \backslash SL(2, \mathbb{R}) / V'$ . We can see this by explicitly looking at how this geodesic lifts to  $O(2) \backslash SL(2, \mathbb{R}) / \tilde{V}'$ . We observe that the code of this geodesic satisfies either the rule that  $n_k = n_0 + k$  for all  $k \geq 1$  or  $n_k = n_0 - k$  for all  $k \geq 1$ . This geodesic therefore does not satisfy the hypotheses. Therefore by Theorem 29, we know the translation flow on the base surface  $A(S, \alpha)$  is uniquely ergodic.

We now recall that the subgroup  $\tilde{V}' \subset V'$  contains derivatives of affine automorphisms of any double cover  $(\tilde{S}_h, \tilde{\alpha}_h)$ . Thus we get ergodicity of the translation flow on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  whenever  $g^t A\tilde{V}'$  is non-divergent in  $SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$ . By Corollary 3, we actually get unique ergodicity of this translation flow.

Because the non-divergent geodesic  $g^t AV'$  is not asymptotic to the convex core, we know that it must eventually be contained in the interior of the convex core. Let  $N \in \mathbb{Z}$  be the integer so that there are infinitely many  $k$  with  $n_k = N$ . Thus we know that there is a  $K$  so that  $k > K$  and  $n_k = N$  implies that the portion of the geodesic  $g^t A\tilde{V}'$  with  $t_k < t < t_{k+1}$  is contained in the portion of the region  $F_N$  which projects to the convex core of  $O(2) \backslash SL(2, \mathbb{R}) / V'$  under the covering map. Let  $\tilde{C}_N \subset F_N$  denote this subset. The region  $F_N$  contains a cusp labeled  $\infty_N$  in Figure 5. Let  $\tilde{D}_N \subset F_N$  be an (quotient of a) horodisk surrounding this cusp which is positive distance from the two geodesic boundary components of  $F_N$ . Every geodesic which enters  $\tilde{D}_N$  must also leave. Observe that for every  $k > K$  with  $n_k = N$ , there is a  $t$  with  $t_k < t < t_{k+1}$  so that the point  $p_k = g^t A\tilde{V}' \in F_N$  lies in  $\tilde{C}_N \setminus \tilde{D}_N$ . Since the set  $\tilde{C}_N \setminus \tilde{D}_N \subset F_N$  is compact, this sequence of points has a convergent subsequence. Therefore,  $g^t A\tilde{V}'$  is non-divergent, and the translation flow on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  is uniquely ergodic by the previous paragraph.  $\square$

We will now begin to consider the cases of statements (b) and (c), where no integer appears in the sequence  $\langle n_k \rangle$  infinitely often. We will make use of the following simplification:

**Remark 36.** *If  $\langle n_k \rangle$  is divergent, then  $\langle n_k \rangle$  diverges to  $s\infty$  as  $k \rightarrow \infty$ , where  $s = 1$  or  $s = -1$  is a choice of the sign. By applying the symmetry  $D(\psi)$  to the geodesic under consideration, we transform the sequence  $\langle n_k \rangle$  into  $\langle -n_k \rangle$ . If we also apply  $\psi_*$  to the element in  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  which determines the cover, the transformation respects both statements and has the effect of toggling the sign  $s$ . Therefore, we can assume without loss of generality that the sequence  $\{n_k\}$  tends to  $+\infty$ .*

We will now explain one special case of statements (b) and (c).

**Lemma 37.** *In the context of the theorem above, suppose that  $\langle n_k \rangle$  limits to  $+\infty$  and that the sequence  $\tau_*^m(h) \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  defined for integers  $m \geq 0$  has an accumulation point in  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  other than  $\mathbf{0}$ . Then, the translation flow is ergodic on  $A(\tilde{S}_h, \tilde{\alpha}_h)$ .*

Again, we note that we get unique ergodicity for the translation flow on the cover in most cases. See Remark 35.

*Proof.* In this case, while we don't get non-divergence of the geodesic  $g^t A\tilde{V}'$  in  $SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$ , we do get non-divergence inside the space of affine deformations of double covers,  $\tilde{\mathcal{O}}_2(S, \alpha)$ .

First of all, let  $h' \neq \mathbf{0}$  be an accumulation point of  $\tau_*^m(h)$ . Observe that the double cover associated to  $h'$  is connected. Let  $m_j$  be an increasing sequence of integers so that  $\tau_*^{m_j}(h)$  converges to  $h'$ . Since  $\langle n_k \rangle$  is a walk on the integers converging to  $+\infty$ , there is a  $J$  so that



for each  $j > J$  there is a  $k = k(j)$  so that  $n_k = m_j$ . Choose such a function  $j \mapsto k(j)$  once and for all.

We will consider the convex core  $C$  within the surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$ , and let  $D$  be a small quotient of a horodisk neighborhood of the cusp. For each integer  $n$ , let  $\tilde{C}_n$  and  $\tilde{D}_n$  be the subsets of the region  $F_n$  consisting of the preimages of  $C$  and  $D$  under the covering map from  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$ , respectively. Then each  $\tilde{C}_n \setminus \tilde{D}_n$  has compact closure. Let  $K_n$  be the closed 1-neighborhood of this closure, which is also compact. Since the geodesic  $g^t AV'$  is non-divergent, it is eventually contained within the convex core. Therefore, by possibly increasing  $J$ , for each  $j > J$ , we can choose a time  $t_j$  with  $t_{n_{k(j)}} < t_j < t_{n_{k(j)+1}}$  so that  $g^{t_j} AV'$  lies in  $K_n$ . We can think of the region  $F_0$  as a concrete subset the hyperbolic plane consisting of two adjacent copies of fundamental domain shown in Figure 4. Then for each  $j > J$ , we can choose an element  $R_j \in V'$  so that

$$g^{t_j} A \tilde{V}' R_j^{-1} = g^{t_j} A R_j^{-1} \tilde{V}' \in F_0.$$

(To get equality here, we are using normality of  $\tilde{V}'$ .) Because  $F_0$  consists of two copies of the fundamental domain, we can also insist that  $\delta(R_j) = \tau^{m_j} \in \text{Isom}(\mathbb{Z})$  is a translation.

Now we will check that we get recurrence in  $\tilde{\mathcal{O}}_2(S, \alpha)$ . We observe that the action of  $R_j$  on the space of covers is the same as the action by translation by  $-m$ . Thus,

$$g^{t_j} A(\tilde{S}_h, \tilde{\alpha}_h) = g^{t_j} A R_j^{-1} R_j(\tilde{S}_h, \tilde{\alpha}_h).$$

The action of  $R_j$  on the space of covers only depends on its image in  $\text{Isom}(\mathbb{Z})$  and

$$R_j(\tilde{S}_h, \tilde{\alpha}_h) = (\tilde{S}_{h_j}, \tilde{\alpha}_{h_j}) \quad \text{where } h_j = \tau_*^{m_j}(h),$$

where equality here is translation equivalence. Thus, we see that  $g^{t_j} A(\tilde{S}_h, \tilde{\alpha}_h)$  is translation equivalent to

$$g^{t_j} A R_j^{-1}(\tilde{S}_{h_j}, \tilde{\alpha}_{h_j}).$$

Now recall that  $\tau_*^{m_j}(h)$  converges to  $h'$ . Also because each  $g^{t_j} A R_j^{-1}$  lies within the compact set  $K_0 \subset F_0$ , by passing to a further subsequence, we can make these matrices converge within  $SL(2, \mathbb{R})$ . This ensures convergence to a connected surface in  $\tilde{\mathcal{O}}_2(S, \alpha)$ , which is a topological quotient of  $SL(2, \mathbb{R}) \times \text{Hom}(\Gamma, \mathbb{Z}_2)$ .  $\square$

The above lemma leaves open the case when the coding walk  $\langle n_k \rangle$  converges to  $+\infty$  and the sequence of homomorphisms  $\tau_*^m(h) \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  converges to the trivial homomorphism,  $\mathbf{0}$ . In order to prove our results about this case, we will need to understand how  $\tau_*^m(h)$  decays to  $\mathbf{0}$ .

A formula for  $\tau_*^1$  is given below.

**Proposition 38.** *The action of translation by one on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  is given by*

$$(\tau_*^1(h))(i) = \begin{cases} h(-i) + h(i-1) + h(i) + h(i+1) & \text{if } i < -2, \\ h(-i) + h(-2) + h(-3) & \text{if } i = -2, \\ h(-i) + h(-2) & \text{if } i = -1, \\ h(-i) & \text{if } i > 0. \end{cases}$$

The action of translation by negative one is given by:

$$(\tau_*^{-1}(h))(i) = \begin{cases} h(-i) & \text{if } i < 0, \\ h(-i) + h(2) & \text{if } i = 1, \\ h(-i) + h(2) + h(3) & \text{if } i = 2, \\ h(-i) + h(i-1) + h(i) + h(i+1) & \text{if } i > 2. \end{cases}$$

*Proof.* This follows from the fact that  $\tau^1 = \delta(\rho \circ \psi)$  and  $\tau^{-1} = \delta(\psi \circ \rho)$ ; see equation 19. The actions of  $\psi$  and  $\rho$  on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  are given in equations 16 and 17, respectively.  $\square$

Now we want to analyze the case when

$$\lim_{m \rightarrow +\infty} \tau_*^m(h) = \mathbf{0}.$$

In order to understand this we introduce the *proximity function* which measures how close a homomorphism is to the trivial homomorphism  $\mathbf{0}$ :

$$(20) \quad P : \text{Hom}(\Gamma, \mathbb{Z}_2) \rightarrow \{1, 2, 3, \dots, \infty\}; \quad P(h) = \begin{cases} \infty & \text{if } h = \mathbf{0}, \\ \min\{|i| : h(i) \neq 0\} & \text{otherwise.} \end{cases}$$

Observe a sequence  $h_n$  tends to  $\mathbf{0}$  if and only if  $P(h_n) \rightarrow \infty$ .

For each integer  $k \geq 0$ , we define a cylinder set in  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  by

$$C_k = \{h : h(k) = h(-k-1) = 1 \text{ and } h(i) = 0 \text{ when } -k-1 < i < k\}.$$

Note that  $h \in C_k$  implies  $P(h) = k$ . These cylinder sets turn out to be very important, for understanding which elements of  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  are limit to the trivial homomorphism  $\mathbf{0}$  under translation.

**Theorem 39.** *The collection of  $h \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  so that  $\lim_{n \rightarrow \infty} \tau_*^n(h) = \mathbf{0}$  is given by*

$$\{\mathbf{0}\} \cup \bigcup_{j \geq 0} \tau_*^{-j}(Z) \quad \text{where} \quad Z = \bigcup_{k \geq 2} \bigcap_{m \geq 0} \tau_*^{-m}(C_{k+m}).$$

Furthermore, for any function  $f : \{i : i \geq 2\} \rightarrow \mathbb{Z}_2$  which is not identically zero, there is an  $h \in Z$  so that  $h(i) = f(i)$  for all  $i \geq 2$ .

The second statement says that there are a number of elements of  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  which limit on  $\mathbf{0}$ . (Informally,  $Z$  has half the information entropy of  $\text{Hom}(\Gamma, \mathbb{Z}_2)$ , or with an appropriate natural metric,  $Z$  has half the Hausdorff dimension of  $\text{Hom}(\Gamma, \mathbb{Z}_2)$ .)

The following lemma is the main ingredient in the proof of the theorem above.

**Lemma 40.** (1) *If  $k \geq 2$  and  $h \in C_k$ , then  $P(\tau_*^1(h)) = k + 1$ .*

(2) *If  $k \geq 2$  is an integer, then  $\tau_*^{-1}(C_{k+1}) \subset C_k$ .*

(3) *If  $k \geq 2$ ,  $P(h) = k$  and  $h \notin C_k$ , then either  $P(\tau_*^1(h)) = k - 1$  and  $\tau_*^1(h) \notin C_{k-1}$  or  $P(\tau_*^2(h)) = k - 1$  and  $\tau_*^2(h) \notin C_{k-1}$ .*

*Proof.* We prove each of the statements below.

(1) Suppose that  $h \in C_k$  and  $k \geq 2$ . Then  $h(k) = h(-k-1) = 1$  while  $h(i) = 0$  for  $-k \leq i \leq k-1$ . Then  $P(\tau_*^1(h)) \leq k+1$ , because  $\tau_*^1(h)(k+1) = h(-k-1) = 1$  by Proposition 38. For  $0 < i \leq k$ , we have  $\tau_*^1(h)(i) = h(-i) = 0$ . Also observe that

$$\tau_*^1(h)(-k) = h(k) + h(-k-1) = 1 + 1 = 0,$$

while the other terms from Proposition 38 vanish. Finally, all terms in the expression for  $\tau_*^1(h)(i)$  vanish when  $-k < i < 0$ . Therefore,  $P(\tau_*^1(h)) = k + 1$  as claimed.

(2) Suppose  $h \in C_{k+1}$  with  $k \geq 2$ . Then  $h(k+1) = h(-k-2) = 1$  while  $h(i) = 0$  for  $-k-1 \leq i \leq k$ . By Proposition 38, we see that  $\tau_*^{-1}(h)(-k-1) = h(k+1) = 1$ . Also,

$$\tau_*^{-1}(h)(k) = \begin{cases} h(-k) + h(2) + h(3) = 0 + 0 + 1 = 1 & \text{if } k = 2, \\ h(-k) + h(k-1) + h(k) + h(k+1) = 0 + 0 + 0 + 1 = 1 & \text{if } k > 2. \end{cases}$$

Now suppose  $-k \leq i < 0$ . Here we have  $\tau_*^{-1}(h)(i) = h(-i) = 0$ . Finally consider the case when  $0 < i < k$ . All terms vanish from the expression for the expression  $\tau_*^{-1}(h)(i)$  given in Proposition 38, so  $\tau_*^{-1}(h)(i) = 0$ . Taken together, we see  $\tau_*^{-1}(h) \in C_k$ .

(3) Suppose  $k \geq 2$ ,  $P(h) = k$  and  $h \notin C_k$ . Since  $P(h) = k$ , we know that  $h(i) = 0$  when  $|i| < k$ . Observe that the set of  $h$  with  $P(h) = k$  and  $h \notin C_k$  is the union of the two pieces:

$$A = \{h : P(h) = k \text{ and } h(-k) = 1\},$$

$$B = \{h : P(h) = k, h(-k) = 0, h(k) = 1, \text{ and } h(-k-1) = 0\}.$$

First assume that  $h \in A$  so that  $h(-k) = 1$  but  $h(i) = 0$  when  $|i| < k$ . Then,  $\tau_*^1(h)(-k+1)$  is given by one of the expressions

$$\begin{cases} h(k-1) + h(-k) + h(-k+1) + h(-k+2) = 0 + 1 + 0 + 0 = 1 & \text{if } k > 3, \\ h(k-1) + h(-2) + h(-3) = 0 + 0 + 1 = 1 & \text{if } k = 3, \\ h(k-1) + h(-2) = 0 + 1 = 1 & \text{if } k = 2. \end{cases}$$

Since proximity can decrease by at most one when applying  $\tau_*^1$ , we see  $P(\tau_*^1(h)) = k-1$ . Also, we have by definition of  $C_{k-1}$  that  $\tau_*^1(h) \notin C_{k-1}$ . Now consider the case when  $h \in B$ . Then  $h(i) = 0$  when  $-k-1 \leq i < k$  and  $h(k) = 1$ . In this case  $\tau_*^1(h)(i) = 0$  when  $|i| < k-1$  since proximity can decrease by at most one. We observe  $\tau_*^1(h)(k-1) = h(-k+1) = 0$ . All terms vanish in the expression for  $\tau_*^1(h)(-k+1)$  given by Proposition 38, so  $\tau_*^1(h)(-k+1) = 0$ . On the other hand,

$$\tau_*^1(h)(-k) = \begin{cases} h(k) + h(-k-1) + h(-k) + h(-k+1) = 1 + 0 + 0 + 0 = 1 & \text{if } k > 2, \\ h(k) + h(-2) + h(-3) = 1 + 0 + 0 = 1 & \text{if } k = 2. \end{cases}$$

Since  $\tau_*^1(h)(-k) = 1$ , we see that  $\tau_*^1(h) \in A$ . Because of our discussion of what happens for elements of  $A$ , we see that  $P(\tau_*^2(h)) = k-1$  and  $\tau_*^2(h) \notin C_{k-1}$ .  $\square$

In the proof of the theorem, it is useful to note the following Corollary to the Lemma above.

**Corollary 41.** *Let  $k \geq 1$  be an integer. If  $P(h) = k$  but  $h \notin C_k$ , then there is an  $n \geq 0$  so that  $P(\tau_*^n(h)) = 1$ .*

*Proof.* We prove this by induction in  $k$ . This holds with  $n = 0$  when  $k = 1$ . When  $k = 2$ , this follows from statement (3) of Lemma 40. Now let  $k \geq 3$  and suppose the conclusion holds when  $P(h) = k-1$  and  $h \notin C_{k-1}$ . Suppose  $P(h) = k$  and  $h \notin C_k$ . Again using statement (3) of Lemma 40, we see that either  $P(\tau_*^1(h)) = k-1$  and  $\tau_*^1(h) \notin C_{k-1}$  or  $P(\tau_*^2(h)) = k-1$  and  $\tau_*^2(h) \notin C_{k-1}$ . Using our induction hypothesis applied to these cases, we see the conclusion holds, and the whole statement holds by induction.  $\square$

*Proof of Theorem 39.* Let  $Z = \bigcup_{k \geq 2} \bigcap_{m \geq 0} \tau_*^{-m}(C_{k+m})$  as in the theorem. Let  $h \in Z$ . Then there is a  $k \geq 2$  so that  $\tau_*^m(h) \in C_{k+m}$  for all  $m \geq 0$ . Then  $P(\tau_*^m(h)) = k + m$  tends to  $\infty$  as  $m \rightarrow \infty$ . Thus,  $\lim_{m \rightarrow \infty} \tau_*^m(h) = \mathbf{0}$ . From this it follows that everything in the set listed in the theorem,  $\{\mathbf{0}\} \cup \bigcup_{j \geq 0} \tau_*^{-j}(Z)$ , limits to  $\mathbf{0}$ .

Conversely, suppose  $h \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  satisfies  $\lim_{n \rightarrow \infty} \tau_*^n(h) = \mathbf{0}$  and  $h \neq \mathbf{0}$ . We know that  $P(\tau_*^n(h)) \rightarrow \infty$  as  $n \rightarrow \infty$ . In particular, we see that for any  $k \geq 1$ , there are only finitely many  $n$  so that

$$(21) \quad P(\tau_*^n(h)) = k.$$

Let  $K \geq 2$  be the smallest value of  $k$  larger than one so that the above equation has a solution for  $n$ , and let  $N$  be the maximal  $n$  satisfying the equation when  $k = K$ . We claim that

$$(22) \quad \tau_*^N(h) \in \bigcap_{n \geq 0} \tau_*^{-n}(C_{K+n}).$$

Otherwise, there is a smallest  $n \geq 0$  so that  $\tau_*^{N+n}(h) \notin C_{K+n}$ . Then by the corollary above, we see that there is an  $m \geq N + n$  so that  $P(\tau_*^m(h)) = 1$ . But since  $P(\tau_*^n(h))$  tends to  $\infty$  and as  $n$  increases the proximity can increase by at most one, we see that there is an  $m' > m$  so that equation 21 is satisfied for  $n = m'$ . But this contradicts the definition of  $N$ . Therefore, equation 22 is true after all. Observe that the equation implies that  $\tau_*^N(h) \in Z$ , and thus  $h \in \bigcup_{j \geq 0} \tau_*^{-j}(Z)$  as desired.

It remains to prove the last sentence of the theorem which guarantees there are a lot of  $h$  so that  $\lim_{n \rightarrow \infty} \tau_*^n(h) = \mathbf{0}$ . A main point here is that for each  $k \geq 2$ , the set  $A_k = \bigcap_{m \geq 0} \tau_*^{-m}(C_{k+m})$  is non-empty. Indeed, as continuous images of cylinder sets, each  $\tau_*^{-m}(C_{k+m})$  is compact. The sets being intersected are nested in the sense that

$$\tau_*^{-m-1}(C_{k+m+1}) \subset \tau_*^{-m}(C_{k+m})$$

by statement (2) of the lemma. Therefore, we can make a choice of an  $\tilde{h}_k \in A_k$  for every  $k \geq 2$ .

Now suppose that  $h_* : \{i \in \mathbb{Z} : i \geq 2\} \rightarrow \mathbb{Z}_2$  is defined and not identically zero. Let  $k = \min\{i : h_*(i) \neq 0\}$ . We will inductively define a sequence of functions  $h_j \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  for  $j \geq k$  so that the sequence converges to an extension of  $h_*$ . Our functions will all lie in  $A_k$ . Since  $A_k$  is closed, this will suffice to prove that the limiting function lies in  $A_k$ . We will also ensure that

$$(23) \quad h_j(i) = h_*(i) \quad \text{when } 2 \leq i \leq j.$$

To ensure convergence of the sequence  $h_j$ , we will also have that

$$(24) \quad h_j(i) = h_{j-1}(i) \quad \text{when } |i| < j.$$

Observe that  $h_*(k) = 1$ , thus we can define  $h_k = \tilde{h}_k \in A_k$ . This serves as our base case. Now assume that  $j > k$  and  $h_{j-1}$  is defined and satisfies the hypotheses above. We will define  $h_j$ . If  $h_{j-1}(j) = h_*(j)$ , then we can take  $h_j = h_{j-1}$ . Otherwise, we define  $h_j = h_{j-1} + \tilde{h}_j$ . Since  $\tilde{h}_j \in C_j$ , we know that  $\tilde{h}_j(j) = 1$ . Therefore, we must have  $h_j(j) = h_*(j)$ . When  $|i| < j$ , we have  $\tilde{h}_j(i) = 0$  because  $h_j \in C_j$ , therefore

$$h_j(i) = h_{j-1}(i) \quad \text{when } |i| < j$$

by inductive hypothesis. This simultaneously ensures both equation (24) holds and verifies equation (23) for  $i < j$  since by hypothesis  $h_{j-1}(i) = h_*(i)$  when  $2 \leq i \leq j-1$ . From the inductive hypothesis and definition of  $\tilde{h}_j$ , we have

$$\tau_*^m(h_{j-1}) \in C_{k+m} \quad \text{and} \quad \tau_*^m(\tilde{h}_j) \in C_{j+m}$$

for each integer  $m \geq 0$ . Observe that when  $j > k$ , the sum of an element in  $C_{k+m}$  and  $C_{j+m}$  lies in  $C_{k+m}$ . Therefore, we see by linearity of  $\tau_*^m$  that

$$\tau_*^m(h_j) = \tau_*^m(h_{j-1}) + \tau_*^m(\tilde{h}_j) \in C_{k+m}$$

for every integer  $m \geq 0$ . Thus,  $h_j \in A_k$  as desired. This completes the inductive step.  $\square$

*Proof of statement (b) of Theorem 34.* We will consider the translation flow on the affine image of a double cover  $A(\tilde{S}_h, \tilde{\alpha}_h)$ , where  $h \in \text{Hom}(\Gamma, \mathbb{Z}_2)$ . From the geodesic  $g^t A\tilde{V}'$ , we define the coding walk  $\{n_k\}$ . We assume that the number of visits to any integer is finite, but that the growth exponent of the visit count satisfies  $v > \varphi^2$ . We assume as allowed by Remark 36 that  $\lim_{k \rightarrow \infty} n_k = +\infty$ . Lemma 37 handles the case when  $\tau_*^m(h)$  does not limit on  $\mathbf{0}$  as  $m \rightarrow +\infty$ . So, we will assume here that  $\lim_{m \rightarrow \infty} \tau_*^m(h) = \mathbf{0}$ .

We will prove ergodicity holds for the translation flow on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  by appealing to Theorem 18. We normalize area so that all double covers have area one. The theorem provides us with ergodicity if for every  $\eta \in (0, 1)$ , the integral of some geometric quantity is infinite. This geometric quantity involves choosing surfaces in the affine deformations of the surface associated to the geodesic.

Fix  $\eta > 0$ . We will now begin to explain how we choose surfaces in  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  whose total area exceeds  $1 - \eta$ . We will make use of the symmetries available to us. After applying a translation to  $SL(2, \mathbb{R})/\tilde{V}'$ , we may assume that our surface lies within the region  $F_0$  in the surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R})/\tilde{V}'$ .

Consider the region  $F_0$  in the surface  $O(2) \backslash SL_{\pm}(2, \mathbb{R})/\tilde{V}'$ . We note that by assumption in statement (b), the geodesic  $g^t A\tilde{V}'$  eventually lies within the interior of the convex core of  $O(2) \backslash SL_{\pm}(2, \mathbb{R})/\tilde{V}'$ . (That it is asymptotic to the convex core is guaranteed by non-divergence of the geodesic, and the geodesic in the boundary has a linearly growing coding walk by Remark 35 which is disallowed by statement (b).) Following the proof of statement (a), we let  $\tilde{C}_0 \subset F_0$  be the preimage of the convex core in  $O(2) \backslash SL_{\pm}(2, \mathbb{R})/\tilde{V}'$ , and we choose an open horodisk  $\tilde{D}_0$  around the cusp in  $F_0$  whose boundary does not touch the boundary of  $\tilde{C}_0$ . Consider a geodesic  $g^t A\tilde{V}'$  passing through  $F_0$  while  $t$  is within the interval  $I \subset \mathbb{R}$ . The region  $\tilde{C}_0 \setminus \tilde{D}_0$  is compact, and has concave boundary, so we see there is a uniform lower bound  $T > 0$  on the Lebesgue measure of those  $t \in I$  so that  $g^t A\tilde{V}'$  lies within the  $\tilde{C}_0 \setminus \tilde{D}_0$ . We can normalize so that the equivalence class of the identity lies within the region  $F_0$ . Then we can choose a subset  $K \subset SL(2, \mathbb{R})$  with compact closure  $\bar{K}$  so that the quotient map

$$O(2) \backslash SL_{\pm}(2, \mathbb{R}) \rightarrow O(2) \backslash SL(2, \mathbb{R})/\tilde{V}'$$

restricted to  $K$  is a bijection to  $\tilde{C}_0 \setminus \tilde{D}_0$ .

When we select our subsurfaces for contribution to the integral, we will always choose two subsurfaces. We will explain first for a double cover of the form  $(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  without an affine change. We think of  $(S, \alpha)$  as depicted by Figure 3: the surface is a topological disk in the plane with edge identifications. For small  $\kappa > 0$ , consider the subset  $U_0(\kappa)$  of this disk consisting of points whose distance from the boundary of the disk is greater than  $\kappa$ . Then

let  $U(\kappa) \subset U_0(\kappa)$  be the subsurface of the largest area. Since these regions exhaust the disk as  $\kappa \rightarrow 0$ , we can choose a  $\kappa$  so that  $U = U(\kappa)$  contains more than a factor of  $1 - \eta$  of the disk's area. We can think of  $U$  as lying in the surface  $(S, \alpha)$ . Our two subsurfaces of a double cover  $(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  will be the two lifts of  $U$ , which take up an area of at least  $1 - \eta$ .

Now consider a surface of the form  $B(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  where  $B \in K \subset SL(2, \mathbb{R})$  is taken so that  $O(2)B\tilde{V}'$  lies within  $\tilde{C}_0 \setminus \tilde{D}_0$ . Here, we just take the images of the subsurfaces of  $(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  described in the previous paragraph under  $B$ .

Theorem 18 specifies a function which we will show has infinite integral. We will give a lower bound on the value of the function for a surface of the form  $B(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  with  $B \in K$ . The distance minimal distance from our subsurfaces to the singularities (boundary of surface) contributes to this function. We observe that because  $K$  has compact closure, there is a positive lower bound,  $\epsilon$ , which is uniform in the choice of  $B \in K$ , for the minimal distance from a point in our subsurfaces of  $B(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  to the singularities of this surface. Similarly, there is an uniform upper bound  $\mathcal{D}$  on the diameters of our two subsurfaces. Note that these quantities do not depend on  $h_* \in \text{Hom}(\Gamma, \mathbb{Z}_2)$ . Finally, we need to consider the maximum over all curves joining our two subsurfaces of the minimum distance from a point on the curve to a singularity. We will choose a canonical curve which depends mostly on the proximity of  $h_*$  to zero,  $P(h_*)$ , which was defined in equation (20). We first consider the case when  $B$  is the identity, so that our double cover consists of two copies of the disk in the plane with identifications. The first time the two pieces are glued to each other is along one of the edges labeled  $p = \pm P(h_*)$  in Figure 3. We choose a curve to leave the first subsurface and move upward along the slope 1 line of symmetry until it reaches the height of the edge  $p$ , then it changes trajectory by  $\pm 45^\circ$  and travels through the midpoint of the edge connecting to the other disk. The curve continues until it hits the line of symmetry in the second disk, and it returns to the second subsurface along the line of symmetry of the disk. It should be observed that there are positive constants  $a$  and  $b$  so that the distance from this curve to the singularities is of the form

$$\min \{a, b\varphi^{-|P(h_*)|}\}.$$

(Because of the self-similarity of the surface, the quantity decays like the scaling constant.) For values of  $B \in K$  other than the identity, we obtain our curve by pushing forward our curve for the identity under  $B$ . This has the effect of changing the constant above by no more than the operator norm of  $B$ , which can be bounded uniformly from above since  $K$  has compact closure. In summary, there are positive constants  $a$  and  $b$  and a constant  $c = 2\mathcal{D}\epsilon^{-2}$ , so that for surfaces of the form  $B(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  with  $B \in K$  the contribution to the integral is greater than

$$\left(c + \frac{1}{\min \{a, b\varphi^{-|P(h_*)|}\}}\right)^{-2}.$$

Now we will estimate the contribution to the integral of a pass through the region  $F_0$ . Suppose  $t_0 < t_1$  are start and end times of a visit geodesic segment  $g^t \tilde{A}\tilde{V}'$  traveling through the lifted convex core  $\tilde{C}_0$  in  $F_0$ . Let  $I \subset (t_0, t_1)$  be the subset of times for which  $g^t \tilde{A}\tilde{V}' \in \tilde{C}_0 \setminus \tilde{D}_0$ . Recall that the Lebesgue measure of  $I$  is at least  $T$ . Then, the contribution to the integral of the portion of the trajectory  $g^t(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$  with  $t_0 < t < t_1$  is at least

$$T \left(c + \frac{1}{\min \{a, b\varphi^{-|P(h_*)|}\}}\right)^{-2}.$$

Now we will return to our original surface and consider the integral over the whole trajectory. By remarks above, we shift time to assume that the geodesic  $g^t AV'$  is within the convex core for  $t > 0$ . Recall the definition of the coding walk  $\{n_k\}$ . For each  $k$ , there are times  $t_k < t_{k+1}$  so that  $g^t \tilde{A}V'$  lies in the region  $F_{n_k}$  when  $t_k < t < t_{k+1}$ . Consider an element  $R \in V'$  so that  $F_{n_k} R^{-1} = F_0$ . Then  $\delta(R)$  is the translation by  $\tau^{-n_k}$ . The surfaces

$$g^t A(\tilde{S}_h, \tilde{\alpha}_h) \quad \text{and} \quad g^t A R^{-1}(\tilde{S}_{h_k}, \tilde{\alpha}_{h_k}),$$

where  $h_k = \tau_*^{n_k}(h)$ , are translation equivalent. In particular these surfaces have the same geometry, while the latter surface is passing through the region  $F_0$ . Thus, there is a contribution to the integral of at least

$$T \left( c + \frac{1}{\min \{a, b\varphi^{-|P(h_k)|}\}} \right)^{-2}.$$

Observe that since  $h_k = \tau_*^{n_k}(h)$ , this contribution only depends on  $n_k$ . For  $N \in \mathbb{Z}$  let  $V_N$  be the number of visits of the coding walk to  $N$ ,  $V_N = \#\{k : n_k = N\}$ . Then, the total integral is bounded from below by

$$\sum_{N=0}^{\infty} T V_N \left( c + \frac{1}{\min \{a, b\varphi^{-|P(h_k)|}\}} \right)^{-2}.$$

Now we invoke the hypothesis that  $\lim_{N \rightarrow \infty} \tau_*^N(h) = \mathbf{0}$ . By Theorem 39, we know that there is an  $M > 0$  and a  $k_0 \in \mathbb{Z}$  so that  $\tau_*^N(h)$  lies in the cylinder set  $C_{k_0+N}$  for all integers  $N > M$ . In particular,  $P(\tau_*^N(h)) = k_0 + N$ . Then for sufficiently large values of  $N$ , say  $N \geq N_0$ , we can arrange that when  $n_k = N$ ,

$$\min \{a, b\varphi^{-|P(h_k)|}\} = b\varphi^{-N-k_0}.$$

This allows us to write a lower bound for the integral as

$$\sum_{N=N_0}^{\infty} T V_N \left( c + \frac{1}{b} \varphi^{N+k_0} \right)^{-2}.$$

We recall that  $T$ ,  $c$ ,  $b$  and  $k_0$  are all positive constants. An application of the root test tells us that this series diverges if

$$v = \limsup_{N \rightarrow \infty} V_N > \varphi^2.$$

This was precisely our hypothesis, and Theorem 18 gives us ergodicity.  $\square$

Now we will consider how to obtain non-ergodic covers. We will make use of ideas of Masur and Smillie which first appeared in [MS91, Theorem 2.1]. The criterion developed there for non-ergodicity carries over from the closed surface case to the infinite genus case. We will state the (only slightly different) version from [MT02, Theorem 3.3] in our setting. For the following theorem recall that the *vertical holonomy* of a curve  $\gamma$  in a translation surface is the imaginary part of  $\int_{\gamma} \alpha$ .

**Theorem 42** (Masur-Smillie [MS91]). *Let  $(S, \alpha)$  be a unit area translation surface of possibly infinite topological type. Suppose there is a sequence of directions  $\theta_n$  tending to the horizontal, and a sequence of partitions of the surface into two pieces,  $S = A_n \sqcup B_n$ , so that the common boundary consists of a countable union of line segments. Assume further that the absolute values of the vertical holonomies of the segments sum to  $h_n < \infty$ . Suppose also that:*

- (i)  $\lim_{n \rightarrow \infty} h_n = 0$ .

- (ii) There are constants  $c$  and  $c'$ , so that  $0 < c < \mu(A_n) < c' < 1$  for each  $n$ , where  $\mu$  is Lebesgue measure on  $(S, \alpha)$ .
- (iii)  $\sum_{n=1}^{\infty} \mu(A_n \Delta A_{n+1}) < \infty$ , where  $\Delta$  denotes symmetric difference.

Then, the translation flow (horizontal straight-line flow) on  $(S, \alpha)$  is not ergodic.

The proof as provided in [MT02, Theorem 3.3] works in our setting, so we omit the proof.

*Proof of statement (c) of Theorem 34.* Let  $A \in SL(2, \mathbb{R})$  determine a non-divergent geodesic  $g^t A V'$  in  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$ . We consider the geodesic  $g^t \tilde{A} \tilde{V}'$  in  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$ , and consider its coding walk  $\{n_k\}$ . Statement (c) concerns the case when the coding walk diverges quickly in the sense that the growth exponent  $v$  is smaller than  $\varphi^2$ . In light of remark 36, we can assume that  $\lim_{k \rightarrow \infty} n_k = +\infty$ .

First, suppose the translation flow on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  is not ergodic. Theorem 1 implies that the only accumulation points of  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  within the space  $\tilde{O}_2(S, \alpha)$  are disconnected. Note that the only disconnected double cover of  $(S, \alpha)$  is the cover determined by the trivial monodromy homomorphism,  $\mathbf{0}$ . Thus, all accumulation points of  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  must be of the form  $B(\tilde{S}_0, \tilde{\alpha}_0) \in \tilde{O}_2(S, \alpha)$  for some  $B \in SL(2, \mathbb{R})$ . We recall that  $\tilde{O}_2(S, \alpha)$  is a topological quotient of  $SL(2, \mathbb{R}) \times \text{Cov}_2(S, \alpha)$ . Here, we have  $\text{Cov}_2(S, \alpha) = \text{Hom}(\Gamma, \mathbb{Z}_2)$ . Written in this way, our trajectory  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  is given by (equivalence classes) of the pairs  $(g^t A, h)$ . Recall that the notion of equivalence is given by the action of the Veech group; the equivalence class of  $(g^t A, h)$  is the collection of pairs  $(g^t A R^{-1}, R_*(h))$  taken over all elements of the Veech group. Since  $\{n_k\}$  tends to  $+\infty$  and makes steps of size at most one, we can find a  $M$  so that for integers  $m > M$ , there is a  $k(m)$  so that  $n_{k(m)} = m$ . Fix one  $m$ . There are times  $t_{k(m)} < t_{k(m)+1}$  so that this portion of the trajectory  $g^t \tilde{A} \tilde{V}'$  lies in the region  $F_m$ . We can apply an element  $R_m \in V'$  to bring our trajectory into  $F_0$ , i.e., when  $t_{k(m)} < t < t_{k(m)+1}$ , we have that  $g^t \tilde{A} \tilde{V}' R^{-1} = g^t A R_m^{-1} V'$  lies in  $F_0$ . The trajectory  $g^t A V'$  is asymptotic to the convex core, so there is a distance  $d > 0$ , realizing the maximal distance of the trajectory to the convex core over times  $t \geq 0$ . We can find a time  $t_m$  with  $t_{n_{k(m)}} < t_m < t_{n_{k(m)}+1}$  so that  $g^{t_m} A R_m^{-1} V'$  lies within distance  $d$  of the convex core, and lies outside a fixed horodisk around the cusp. By making the same choices for all  $m$ , we get points  $g^{t_m} A R_m^{-1} \tilde{V}'$  in  $F_0$  taken from a set with compact closure. By possibly changing each  $R_m$  to a different group element in the class  $R_m \tilde{V}'$ , we can assume that  $g^{t_m} A R_m^{-1}$  all lie in a compact subset of  $SL(2, \mathbb{R})$ . Recall that the action of  $R_m$  on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  only depends on  $\delta(R_m) = \tau^m$ . Therefore

$$[g^{t_m} A, h] = [g^{t_m} A R_m^{-1}, \tau_*^m(h)]$$

as elements of  $\tilde{O}_2(S, \alpha)$ . We can now show our desired conclusion that  $\lim_{m \rightarrow \infty} \tau_*^m(h) = \mathbf{0}$ . Suppose this was not true. Then, there would be infinitely many  $m$  so that  $\tau_*^m(h)$  lies outside of some open neighborhood of  $\mathbf{0}$  in  $\text{Cov}_2(S, \alpha)$ . Because  $\text{Cov}_2(S, \alpha)$  is compact, there we can find a sequence  $m_j \rightarrow \infty$  so that  $\tau_*^{m_j}(h)$  converges to some  $h_* \neq \mathbf{0}$ . Up to passing to a further subsequence, we can also assume that  $g^{t_{m_j}} A R_{m_j}^{-1}$  converges to some  $B \in SL(2, \mathbb{R})$  by the compactness noted above. Then,

$$\lim_{j \rightarrow \infty} [g^{t_{m_j}} A, h] = \lim_{j \rightarrow \infty} [g^{t_{m_j}} A R_{m_j}^{-1}, \tau_*^{m_j}(h)] = [B, h_*].$$

But this shows that our trajectory in  $\tilde{O}_2(S, \alpha)$  has an accumulation point representing a connected surface,  $B(\tilde{S}_{h_*}, \tilde{\alpha}_{h_*})$ . This together with our hypothesis of non-ergodicity violates Theorem 1.



Now suppose that  $\lim_{m \rightarrow +\infty} \tau_*^m(h) = \mathbf{0}$ . We will prove that the translation flow on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  is not ergodic. To do this, we will find a sequence of partitions satisfying the criteria set out by Masur and Smillie (Theorem 42). Our partitions will be obtained by pulling back partitions of the deformed surface  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  along a sequence of times tending to  $+\infty$ .

It will be important for us to label the lifts of the basepoint to our double covers. We label the lifts by the elements of  $\mathbb{Z}_2$ . Then when we deform our surfaces or apply linear maps, we will respect the labels of the basepoint. (Note that all double covers are normal and so admit a translation automorphism swapping the lifts of the basepoint.)

We introduce the following setup. Consider the region  $F_0$  in  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$ . Since the geodesic  $g_t AV'$  is asymptotic to the convex core of  $O(2) \backslash SL_{\pm}(2, \mathbb{R}) / V'$ , there is a  $d > 0$  so that this geodesic is contained entirely in the  $d$ -neighborhood of this convex core. Let  $\tilde{C}'_0 \subset F_0$  consist of those points of  $F_0$  which project to this  $d$ -neighborhood of the convex core. Let  $\tilde{D}_0$  be a small horodisk about the cusp in  $F_0$  whose boundary does not touch either of two geodesic boundaries of  $F_0$ . We always normalize so that the equivalence class of the identity,  $I\tilde{V}'$ , lies in  $F_0$ . Now consider another point  $B\tilde{V}'$  of  $F_0$ . Select a path  $\gamma_B$  in  $F_0$  joining  $I\tilde{V}'$  to  $B\tilde{V}'$ . We can lift this path in  $SL_{\pm}(2, \mathbb{R}) / \tilde{V}'$  to a path in  $SL(2, \mathbb{R})$  beginning at the identity. We let  $E(B) \in SL(2, \mathbb{R})$  denote the endpoint of this path. Observe that  $E(B)$  lies in the coset  $B\tilde{V}'$ . The selection of  $E(B)$  is not quite canonical; it does not depend on the choice of  $B$  from the class  $B\tilde{V}'$ , but it depends on the homotopy class of the curve  $\gamma_B$ . Since  $F_0$  is topologically an annulus, a different choice of path might give us a different  $E(B)$ , and difference is explained by monodromy around the cusp. Let  $\Psi = \langle D(\psi)^2 \rangle \subset SL(2, \mathbb{R})$  be the monodromy group of  $F_0$ . We see that while  $E(B)$  is not canonical, the coset  $E(B)\Psi$  is. We will make choices only depending on  $E(B)\Psi$ .

Now consider an affine image of our cover,  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$ , and suppose that  $g^t A\tilde{V}'$  lies in the region  $F_n$ . Select an element  $R \in V'$  so that  $g^t A\tilde{V}'R^{-1} = g^t AR^{-1}\tilde{V}'$  lies in the region  $F_0$ . The action of  $R$  on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$  is given by  $\delta(R) = \tau_*^n$ . Therefore,

$$g^t A(\tilde{S}_h, \tilde{\alpha}_h) = g^t AR^{-1}(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)}),$$

where equality denotes translation equivalence respecting marked points. Now consider  $E(g^t AR^{-1})$  as in the prior paragraph. We have  $E(g^t AR^{-1}) = g^t AR^{-1}\tilde{R}^{-1}$  for some  $\tilde{R} \in \tilde{V}'$ . Since  $\tilde{V}'$  acts trivially on  $\text{Hom}(\Gamma, \mathbb{Z}_2)$ , we see that

$$(25) \quad g^t A(\tilde{S}_h, \tilde{\alpha}_h) = E(g^t AR^{-1})(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)}).$$

A partition of the cover  $(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$  will then pull back to a partition of our original surface  $A(\tilde{S}_h, \tilde{\alpha}_h)$ . The cover  $(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$  can be thought of two copies of the region depicted in Figure 3, labeled  $\mathcal{R}_0$  and  $\mathcal{R}_1$  and glued together according to  $\tau_*^n(h) \in \text{Hom}(\Gamma, \mathbb{Z}_2)$ . The index of the regions is determined by the index of the lift of the basepoint the region contains. We will partition the cover into two subsurfaces  $\mathcal{A}$  and  $\mathcal{B}$ . From our point of view, there are three types of vertical cylinders on the cover  $(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$ :

- (1) vertical cylinders that stay entirely in region  $\mathcal{R}_0$ .
- (2) vertical cylinders that stay entirely in region  $\mathcal{R}_1$ .
- (3) vertical cylinders pass through both  $\mathcal{R}_0$  and  $\mathcal{R}_1$ .

For any integer  $j < 0$ , we have either  $\tau_*^n(h)(j) = 0$  or  $\tau_*^n(h)(j) = 1$ . In the first case, we get vertical cylinders of types (1) and (2) passing through lifts of the edge labeled  $j$  of the base

surface (as depicted in Figure 3), and in the second case, there is a single vertical cylinder of type (3) which passes through both the lifts of edges labeled  $j$ . We define the surface  $\mathcal{A}_n \subset (\tilde{S}_{\tau^n(h)}, \tilde{\alpha}_{\tau^n(h)})$  to be the union of all vertical cylinders of type (1), and we define  $\mathcal{B}_n$  to be the union of cylinders of the remaining two types. We can push this partition onto the surface  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  by applying the affine map  $E(g^t AR^{-1})$  and using the identification given in equation 25. This gives us a partition  $(\mathcal{A}^t, \mathcal{B}^t)$  of  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$ . Note that while  $E(g^t AR^{-1})$  is not quite canonical as noted in the prior paragraph, it is well defined up to a power of  $D(\psi)^2$ . Note that the partition of  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  obtained is the same no matter which element of  $E(g^t AR^{-1})\Psi$  we choose, because the action of  $D(\psi^2) \in \tilde{V}'$  preserves each vertical cylinder.

For each  $t$ , we can pullback the partition  $(\mathcal{A}^t, \mathcal{B}^t)$  of  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$  to a partition  $g^{-t}(\mathcal{A}^t, \mathcal{B}^t)$  of  $A(\tilde{S}_h, \tilde{\alpha}_h)$ . This is actually a sequence of partitions, as we now explain. When  $g^t \tilde{V}'$  lies in the region  $F_n$ , we have that  $t$  lies in some interval  $(t_k, t_{k+1})$  where  $n_k = n$ . We claim the partition  $g^{-t}(\mathcal{A}^t, \mathcal{B}^t)$  is independent of the choice of  $t$  from this interval. This is because the coset  $g^{-t}E(g^t AR^{-1})\Psi$  is constant on the interval, since given a path to  $g^t AR^{-1}$  we can get a path to  $g^{t*} AR^{-1}$  by traveling along the geodesic, and this change is canceled by the application of  $g^{-t}$ . Thus the partition only depends on  $k$ , and we define  $\mathcal{A}'_k = g^{-t}(\mathcal{A}^t)$  and  $\mathcal{B}'_k = g^{-t}(\mathcal{B}^t)$  for any  $t$  satisfying  $t_k < t < t_{k+1}$ . These subsurfaces partition  $A(\tilde{S}_h, \tilde{\alpha}_h)$ .

It remains to check that our sequence of partitions  $(\mathcal{A}'_k, \mathcal{B}'_k)$  satisfy the criterion of Masur and Smillie.

Consider statement (i), we need to show the total vertical holonomy of  $\partial\mathcal{A}'_k$  tends to zero as  $k \rightarrow \infty$ . This can be based on the observation that the total length of the boundary of the partitions  $(\mathcal{A}_n, \mathcal{B}_n)$  of the surface  $(\tilde{S}_{\tau^n(h)}, \tilde{\alpha}_{\tau^n(h)})$  can be bounded from above independent of  $n$  and  $\tau^n(h)$ . Indeed, for any double cover, the total length of all boundaries of all vertical cylinders is a finite constant independent of the cover. It is twice the corresponding constant for the base surface  $(S, \alpha)$ , and the constant there is finite because the cylinders decay in size exponentially. We can choose  $R \in V'$  as above so that  $g^t AR^{-1}\tilde{V}'$  lies in the region  $F_0$  when  $t_k < t < t_{k+1}$ . Select  $t$  so that  $g^t AR^{-1}\tilde{V}'$  lies in a compact set of  $F_0$ , which can be taken to be a neighborhood of the convex core with a horodisk removed as before. Note then that by definition of  $E$ , the quantity  $E(g^t AR^{-1})\Psi$  is taken from a compact subset of  $SL(2, \mathbb{R})/\Psi$ . So, in particular we get a uniform upper bound  $L < \infty$  on the length of the boundary of the partition  $(\mathcal{A}^t, \mathcal{B}^t)$  of the surface  $g^t A(\tilde{S}_h, \tilde{\alpha}_h)$ . (This uses the identification of equation 25.) In particular, the vertical component of this length is bounded from above by  $L$ . Pulling back via  $g^{-t}$ , we see that the vertical component of the length of the common boundary of  $\mathcal{A}'_k$  and  $\mathcal{B}'_k$  is bounded from above by  $e^{-t}L$ , which tends to zero as  $k \rightarrow \infty$  because there is a uniform lower bound on  $t_{k+1} - t_k$ . This verifies statement (i).

To prove statement (ii), we need to make use of the assumptions that  $\lim_{k \rightarrow \infty} n_k = +\infty$  and  $\lim_{m \rightarrow \infty} \tau_*^m(h) = \mathbf{0}$ . By Theorem 39, there is an  $M \geq 0$  and an integer  $j$  so that  $m > M$  implies that  $P(\tau_*^m(h)) = j + m$ , where  $P$  denotes proximity; see equation 20. Since  $\{n_k\}$  tends to  $+\infty$ , there is a  $K$  so that  $k > K$  implies  $P(\tau_*^{n_k}(h)) > 1$ . Since the proximity is larger than one,  $\tau_*^{n_k}(h)(-1) = 0$ . This means that there are two vertical cylinders in the surface  $(\tilde{S}_{\tau_*^{n_k}(h)}, \tilde{\alpha}_{\tau_*^{n_k}(h)})$  passing through lifts of the edge labeled  $-1$ . One of these cylinders lies in  $\mathcal{A}_{n_k}$  and the other lies in  $\mathcal{B}_{n_k}$ . Thus when  $k > K$ , we obtain upper and lower bounds on the area of  $\mathcal{A}_{n_k}$ . Since the partition  $(\mathcal{A}'_k, \mathcal{B}'_k)$  was obtained by pulling back along an area preserving map, the same bounds hold here. Note statement (ii) holds only after passing to the sequence of partitions  $\{(\mathcal{A}'_k, \mathcal{B}'_k) : k > K\}$ , i.e., we drop an initial segment of partitions.

In order to understand (iii), we need to investigate how our partition changes when the geodesic  $g^t A\tilde{V}'$  passes from a region  $F_n$  to  $F_{n+1}$ . Consider a time  $t$  so that  $g^t A\tilde{V}'$  lies in the common boundary between  $F_n$  and  $F_{n+1}$ . This is the point at which the partition changes. First consider this point  $g^t A\tilde{V}'$  as part of  $F_n$ . Then, we build a partition  $(\mathcal{A}_n, \mathcal{B}_n)$  of the cover  $(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$  as described above, and apply  $g^{-t}E(g^t AR_n^{-1})$ , where  $R_n \in V'$  is any Veech group element carrying  $F_0$  to  $F_n$ , to partition our original surface. Here  $E(g^t AR_n^{-1}) \in SL(2, \mathbb{R})$  is determined by lifting a path joining the identity to  $g^t AR_n^{-1}\tilde{V}'$  within  $F_0$ . Now we consider what happens when we consider  $g^t A\tilde{V}'$  as part of  $F_{n+1}$ . We construct a partition  $(\mathcal{A}_{n+1}, \mathcal{B}_{n+1})$  of the cover  $(\tilde{S}_{\tau_*^{n+1}(h)}, \tilde{\alpha}_{\tau_*^{n+1}(h)})$ . We apply  $g^{-t}E(g^t AR_{n+1}^{-1})$ , where  $R_{n+1}$  is some Veech group element carrying  $F_0$  to  $F_{n+1}$ , to partition our original surface. We need to understand the area of the symmetric difference of the pulled back partitions. It is equivalent to find the area of the symmetric difference between the partition  $(\mathcal{A}_n, \mathcal{B}_n)$  and the other partition of  $(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$  obtained as the image of  $(\mathcal{A}_{n+1}, \mathcal{B}_{n+1})$  under

$$R_* = (g^{-t}E(g^t AR_n^{-1}))^{-1}g^{-t}E(g^t AR_{n+1}^{-1}) = E(g^t AR_n^{-1})^{-1}E(g^t AR_{n+1}^{-1}).$$

The elements  $E(g^t AR_n^{-1})$  and  $E(g^t AR_{n+1}^{-1})$  are determined based on lifting paths  $\gamma_n$  and  $\gamma_{n+1}$  in  $SL_{\pm}(2, \mathbb{R})/\tilde{V}'$  to paths  $\tilde{\gamma}_n$  and  $\tilde{\gamma}_{n+1}$  respectively. The value of  $R_* \in SL(2, \mathbb{R})$  can then be determined by lifting the path obtained by first following  $\gamma_1$  and then following the translated path  $E(g^t AR_n^{-1})(\gamma_2)$  backward. The result is a path which passes once through the common boundary between  $F_0$  and  $F_1$ , and joins the equivalence class of the identity to  $D(\rho \circ \psi)\tilde{V}'$ . The value of  $R_*$  is the endpoint of this path lifted to  $SL(2, \mathbb{R})$ , which we see lies in the double coset

$$\Psi D(\rho \circ \psi) \Psi \in \Psi \backslash SL(2, \mathbb{R}) / \Psi.$$

As the action of an element  $\Psi$  does not change our partitions, we can choose to work with the simplest element from our point of view,  $R_* = D(\rho \circ \psi)$ .

We need to estimate the area of the symmetric difference  $\mathcal{A}_n \Delta R_*(\mathcal{A}_{n+1})$ . Here, we interpret  $R_*$  as an affine homeomorphism

$$R_* : (\tilde{S}_{\tau_*^{n+1}(h)}, \tilde{\alpha}_{\tau_*^{n+1}(h)}) \rightarrow (\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$$

which is characterized by its derivative and respects the labels of lifts of the basepoint. Observe that the action of the matrix  $R_*$  carries vertical cylinders to horizontal cylinders and preserves their widths. It also respects the labeling of the lifts of the basepoints. Suppose that the proximity  $P(\tau_*^{n+1}(h)) = p$ . For any integer  $e < 0$  with  $-e < p$ , we have  $\tau_*^{n+1}(h)(e) = 0$ , and thus the vertical cylinder in  $(\tilde{S}_{\tau_*^{n+1}(h)}, \tilde{\alpha}_{\tau_*^{n+1}(h)})$  starting in regions  $\mathcal{R}_0$  and passing through a lift of the edge labeled  $e$  in Figure 3 lies in the subsurface  $\mathcal{A}_{n+1}$ . Consider the union  $U \subset \mathcal{A}$  of all such vertical cylinders in  $\mathcal{A}_{n+1}$  with  $-p < e < 0$ . We observe that because basepoint labels are preserved,  $R_*(U) \subset \mathcal{A}_n$ , with each vertical cylinder through  $e$  being sent to a horizontal cylinder in  $\mathcal{A}_n$  passing through a vertical edge labeled  $-e$ . Let  $\nu$  be the normalized Lebesgue measure on  $(\tilde{S}_{\tau_*^n(h)}, \tilde{\alpha}_{\tau_*^n(h)})$ . We get the upper bound on the area of the symmetric difference

$$\nu(\mathcal{A}_n \Delta R_*(\mathcal{A}_{n+1})) \leq \nu(\mathcal{A}_n) + \nu(R_*(\mathcal{A}_{n+1})) - 2\nu(U) \leq 1 - 2\nu(U).$$

Observe that successively smaller cylinders in  $(S, \alpha)$  decrease in area by a factor of  $\varphi^2$ . It follows that there is a constant  $\alpha > 0$  so that  $1 - 2\nu(U) < \alpha\varphi^{-2p}$ . So, we see

$$\nu(\mathcal{A}_n \Delta R_*(\mathcal{A}_{n+1})) < \alpha\varphi^{-2p},$$

where  $p = P(\tau_*^{n+1}(h))$  as above.

Let  $\mu$  be the normalized Lebesgue measure on the surface  $A(\tilde{S}_h, \tilde{\alpha}_h)$ . The prior paragraph gives an upper bound on  $\mu(\mathcal{A}'_k \Delta \mathcal{A}'_{k+1})$  when  $n_{k+1} = n_k + 1$  in terms of the proximity  $p(n_{k+1}) = P(\tau_*^{n_{k+1}}(h))$ . A similar bound holds for the case when  $n_{k+1} = n_k - 1$ ; there is a constant  $\alpha' > \alpha$  so that

$$\mu(\mathcal{A}'_k \Delta \mathcal{A}'_{k+1}) < \alpha' \varphi^{-2p(n_{k+1})}$$

regardless if  $n_{k+1}$  equals  $n_k + 1$  or  $n_k - 1$ .

Now we will consider the total sum of the symmetric differences to verify statement (iii). Let  $V_N = \#\{k : n_k = N\}$  for integers  $N$ . By assumption, each  $V_N$  is finite, and  $V_N = 0$  for  $N < N_0$  for some  $N_0 \in \mathbb{Z}$ . Then,

$$\sum_k \mu(\mathcal{A}'_k \Delta \mathcal{A}'_{k+1}) \leq \sum_{N=N_0}^{\infty} V_N \varphi^{-2p(N)}.$$

Now we incorporate the assumption that  $\lim_{N \rightarrow \infty} \tau_*^N(h) = \mathbf{0}$ . From Theorem 39, we obtain an  $M \geq 0$  and an integer  $j$  so that  $N > M$  implies that  $P(\tau_*^N(h)) = j + N$ . Let  $X < \infty$  be the sum over the terms with  $N \leq M$ , then we see that

$$\sum_k \mu(\mathcal{A}'_k \Delta \mathcal{A}'_{k+1}) \leq X + \sum_{N=N_0}^{\infty} V_N \varphi^{-2(j+N)}.$$

An application of the root test tells us this sum converges if  $\limsup_{N \rightarrow \infty} V_N^{1/N} < \varphi^2$ . Since this was a hypothesis, we have verified statement (iii). As an application of Theorem 42, we see that the translation flow on  $A(\tilde{S}_h, \tilde{\alpha}_h)$  is not ergodic.  $\square$

## REFERENCES

- [BM74] Alan F. Beardon and Bernard Maskit, *Limit points of Kleinian groups and finite sided fundamental polyhedra*, Acta Math. **132** (1974), 1–12. MR 0333164 (48 #11489)
- [Bow13] Joshua P. Bowman, *The complete family of Arnoux-Yoccoz surfaces*, Geometriae Dedicata **164** (2013), no. 1, 113–130 (English).
- [Buf13] Aleksandr Igorevich Bufetov, *Limit theorems for suspension flows over vershik automorphisms*, Russian Mathematical Surveys **68** (2013), no. 5, 789–860.
- [CGL06] R. Chamanara, F. P. Gardiner, and N. Lakic, *A hyperelliptic realization of the horseshoe and baker maps*, Ergodic Theory Dynam. Systems **26** (2006), no. 6, 1749–1768. MR 2279264 (2008j:37088)
- [Cha04] R. Chamanara, *Affine automorphism groups of surfaces of infinite type*, In the tradition of Ahlfors and Bers, III, Contemp. Math., vol. 355, Amer. Math. Soc., Providence, RI, 2004, pp. 123–145. MR 2145060 (2006b:30077)
- [DEDML98] Mirko Degli Esposti, Gianluigi Del Magno, and Marco Lenci, *An infinite step billiard*, Nonlinearity **11** (1998), no. 4, 991–1013. MR 1632594 (99i:58092)
- [Hoo10] W. Patrick Hooper, *The invariant measures of some infinite interval exchange maps*, to appear in Geometry and Topology, arXiv:1005.1902, 2010.
- [LT14] Kathryn Lindsey and Rodrigo Treviño, *Flat surface models of ergodic systems*, Preprint.
- [Mas92] Howard Masur, *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential.*, Duke Math. J. **66** (1992), no. 3, 387–442 (English), Contains Masur’s Criterion for ergodicity.
- [MS91] Howard Masur and John Smillie, *Hausdorff dimension of sets of nonergodic measured foliations*, Ann. of Math. (2) **134** (1991), no. 3, 455–543. MR 1135877 (92j:58081)

- [MT02] Howard Masur and Serge Tabachnikov, *Rational billiards and flat structures*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 1015–1089. MR 1928530 (2003j:37002)
- [PSV11] Piotr Przytycki, Gabriela Schmithüsen, and Ferrán Valdez, *Veech groups of Loch Ness monsters.*, Ann. Inst. Fourier **61** (2011), no. 2, 673–687 (English).
- [Pud13] Doron Puder, *Primitive words, free factors and measure preservation*, Israel Journal of Mathematics (2013), 1–49 (English).
- [Rya97] Kay Ryan, *Elephant rocks*, Grove Press, 1997.
- [Thu88] William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431. MR 956596 (89k:57023)
- [Tre14] Rodrigo Treviño, *On the ergodicity of flat surfaces of finite area*, Geometric and Functional Analysis (2014), 1–27 (English).
- [Tro99] Serge Troubetzkoy, *Billiards in infinite polygons*, Nonlinearity **12** (1999), no. 3, 513–524. MR 1690190 (2000b:37040)
- [Vee69] W.A. Veech, *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2.*, Trans. Am. Math. Soc. **140** (1969), 1–33 (English).
- [Vee89] W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583. MR 1005006 (91h:58083a)

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